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# Dynamics Theory and Applications

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**Thomas R. Kane**

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**McGraw-Hill Book Company, 1985**

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**McGraw-Hill Book Company**

New York St. Louis San Francisco Auckland Bogotá Hamburg  
Johannesburg London Madrid Mexico Montreal New Delhi  
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

This book was set in Times Roman.  
The editors were Anne Murphy and Madelaine Eichberg;  
the production supervisor was Diane Renda.  
The drawings were done by Wellington Studios Ltd.  
Halliday Lithograph Corporation was printer and binder.

**DYNAMICS:**

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234567890 HALHAL 898765

**ISBN 0-07-037846-0**

**Library of Congress Cataloging in Publication Data**

Kane, Thomas R.

Dynamics, theory and applications.

(McGraw-Hill series in mechanical engineering)

Bibliography: p.

Includes index.

1. Dynamics. I. Levinson, David A. II. Title.

III. Series.

TA352.K36 1985 531'.11 84-21802

ISBN 0-07-037846-0



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## PREFACE

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Dissatisfaction with available textbooks on the subject of dynamics has been widespread throughout the engineering and physics communities for some years among teachers, students, and employers of university graduates; furthermore, this dissatisfaction is growing at the present time. A major reason for this is that engineering graduates entering industry, when asked to solve dynamics problems arising in fields such as multibody spacecraft attitude control, robotics, and design of complex mechanical devices, find that their education in dynamics, based on the textbooks currently in print, has not equipped them adequately to perform the tasks confronting them. Similarly, physics graduates often discover that; in their education, so much emphasis was placed on preparation for the study of quantum mechanics, and the subject of rigid body dynamics was slighted to such an extent, that they are handicapped, both in industry and in academic research, but their inability to design certain types of experimental equipment, such as a particle detector that is to be mounted on a planetary satellite. In this connection, the ability to analyze the effects of detector scanning motions on the attitude motion of the satellite is just as important as knowledge of the physics of the detection process itself. Moreover, the graduates in question often are totally unaware of the deficiencies in their dynamics education. How did this state of affairs come into being, and is there a remedy?

For the most part, traditional dynamics texts deal with the exposition of eighteenth-century methods and their application to physically simple systems, such as the spinning top with a fixed point, the double pendulum, and so forth. The reason for this is that, prior to the advent of computers, one was justified in demanding no more of students than the ability to formulate equations of motion for such simple systems, for one could not hope to extract useful information from the equations governing the motions of more complex systems. Indeed, considerable ingenuity and a rather extensive knowledge of mathematics were required to analyze even simple systems. Not surprisingly, therefore, even more attention came to be focused on analytical intricacies of the *mathematics* of

dynamics, while the process of formulating equations of motion came to be regarded as a rather routine matter. Now that computers enable one to extract highly valuable information from large sets of complicated equations of motion, all this has changed. In fact, the inability to *formulate* equations of motion effectively can be as great a hindrance at present as the inability to *solve* equations was formerly. It follows that the subject of formulation of equations of motion demands careful reconsideration. Or, to say it another way, a major goal of a modern dynamics course must be to produce students who are proficient in the use of the best available methodology for formulating equations of motion. How can this goal be attained?

In the 1970s, when extensive dynamical studies of multibody spacecraft, robotic devices, and complex scientific equipment were first undertaken, it became apparent that straightforward use of classical methods, such as those of Newton, Lagrange, and Hamilton, could entail the expenditure of very large, and at times even prohibitive, amounts of analysts' labor, and could lead to equations of motion so unwieldy as to render computer solutions unacceptably slow for technical and/or economic reasons. Now, while it may be impossible to overcome this difficulty entirely, which is to say that it is unlikely that a way will be found to reduce formulating equations of motion for complex systems to a truly simple task, there does exist a method that is superior to the classical ones in that its use leads to major savings in labor, as well as to simpler equations. Moreover, being highly systematic, this method is easy to teach. Focusing attention on motions, rather than on configurations, it affords the analyst maximum physical insight. Not involving variations, such as those encountered in connection with virtual work, it can be presented at a relatively elementary mathematical level. Furthermore, it enables one to deal directly with nonholonomic systems without having to introduce and subsequently eliminate Lagrange multipliers. It follows that the resolution of the dilemma before us is to instruct students in the use of this method (which is often referred to as Kane's method). This book is intended as the basis for such instruction.

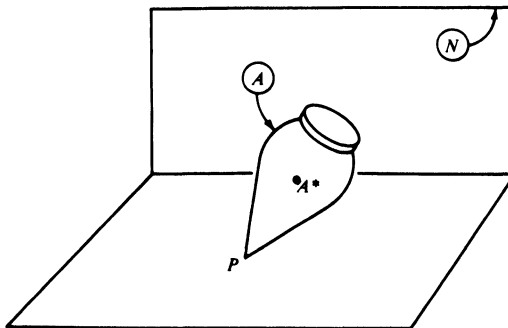
Textbooks can differ from each other not only in content but also in organization, and the sequence in which topics are presented can have a significant effect on the relative ease of teaching and learning the subject. The rationale underlying the organization of the present book is the following. We view dynamics as a deductive discipline, knowledge of which enables one to describe in quantitative and qualitative terms how mechanical systems move when acted upon by given forces, or to determine what forces must be applied to a system in order to cause it to move in a specified manner. The solution of a dynamics problem is carried out in two major steps, the first being the formulation of equations of motion, and the second the extraction of information from these equations. Since the second step cannot be taken fruitfully until the first has been completed, it is imperative that the distinction between the two be kept clearly in mind. In this book, the extraction of information from equations of motion is deferred formally to the last chapter, while the preceding chapters deal with the material one needs to master in order to be able to arrive at valid equations of motion.



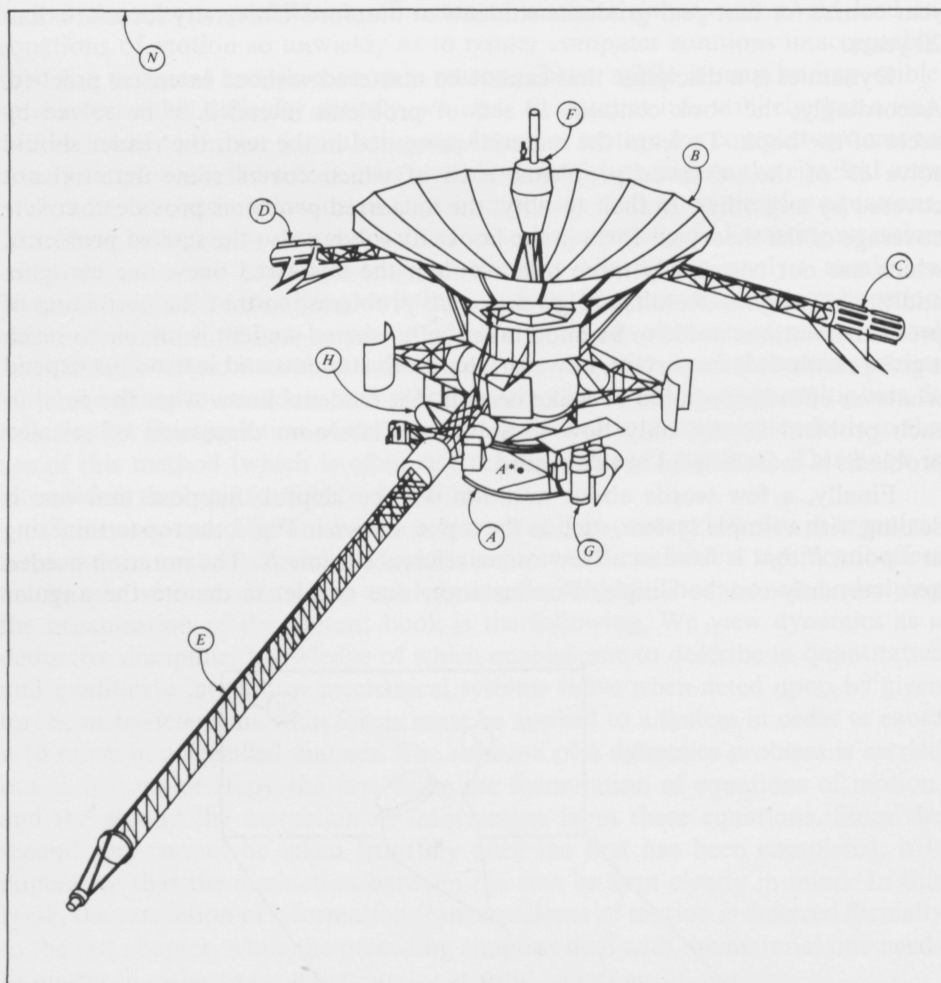
Diverse concepts come into play in the process of constructing equations of motion. Here again it is important to separate ideas from each other distinctly. Major attention must be devoted to kinematics, mass distribution considerations, and force concepts. Accordingly, we treat each of these topics in its own right. First, however, since differentiation of vectors plays a key role in dynamics, we devote the initial chapter of the book to this topic. Here we stress the fact that differentiation of a vector with respect to a scalar variable requires specification of a reference frame, in which connection we dispense with the use of limits because such use tends to confuse rather than clarify matters; but we draw directly on students' knowledge of scalar calculus. Thereafter, we devote one chapter each to the topics of kinematics, mass distribution, and generalized forces, before discussing energy functions, in Chapter 5, and the formulation of equations of motion, in Chapter 6. Finally, the extraction of information from equations of motion is considered in Chapter 7. This material has formed the basis for a one-year course for first-year graduate students at Stanford University for more than 20 years.

Dynamics is a discipline that cannot be mastered without extensive practice. Accordingly, the book contains 14 sets of problems intended to be solved by users of the book. To learn the material presented in the text, the reader should solve *all* of the *unstarred* problems, each of which covers some material not covered by any other. In their totality, the unstarred problems provide complete coverage of the theory set forth in the book. By solving also the starred problems, which are not necessarily more difficult than the unstarred ones, one can gain additional insights. Results are given for *all* problems, so that the correcting of problem solutions needs to be undertaken only when a student is unable to reach a given result. It is important, however, that both students and instructors expend whatever effort is required to make certain that students know what the *point* of each problem is, not only how to solve it. Classroom discussion of selected problems is most helpful in this regard.

Finally, a few words about notation will be helpful. Suppose that one is dealing with a simple system, such as the top *A*, shown in Fig. *i*, the top terminating in a point *P* that is fixed in a Newtonian reference frame *N*. The notation needed here certainly can be simple. For instance, one can let  $\omega$  denote the angular



velocity of  $A$  in  $N$ , and let  $\mathbf{v}$  stand for the velocity in  $N$  of point  $A^*$ , the mass center of  $A$ . Indeed, notations more elaborate than these can be regarded as objectionable because they burden the analyst with unnecessary writing. But suppose that one must undertake the analysis of motions of a complex system, such as the Galileo spacecraft, modeled as consisting of eight rigid bodies  $A, B, \dots, H$ , coupled to each other as indicated in Fig. *ii*. Here, unless one employs notations more elaborate than  $\omega$  and  $\mathbf{v}$ , one cannot distinguish from each other such quantities as, say, the angular velocity of  $A$  in a Newtonian reference frame  $N$ , the angular velocity of  $B$  in  $N$ , and the angular velocity of  $B$  in  $A$ , all of which may enter the analysis. Or, if  $A^*$  and  $B^*$  are points of interest fixed on  $A$  and  $B$ , perhaps the respective mass centers, one needs a notation that permits one to distinguish from each other, say, the velocity of  $A^*$  in  $N$ , the velocity of  $B^*$  in  $N$ , and the velocity of  $B^*$  in  $A$ . Therefore, we establish, and use



consistently throughout this book, a few notational practices that work well in such situations. In particular, when a vector denoting an angular velocity or an angular acceleration of a rigid body in a certain reference frame has two superscripts, the right superscript stands for the rigid body, whereas the left superscript refers to the reference frame. Incidentally, we use the terms “reference frame” and “rigid body” interchangeably. That is, every rigid body can serve as a reference frame, and every reference frame can be regarded as a massless rigid body. Thus, for example, the three angular velocities mentioned in connection with the system depicted in Fig. *ii*, namely, the angular velocity of  $A$  in  $N$ , the angular velocity of  $B$  in  $N$ , and the angular velocity of  $B$  in  $A$ , are denoted by  ${}^N\omega^A$ ,  ${}^N\omega^B$ , and  ${}^A\omega^B$ , respectively. Similarly, the right superscript on a vector denoting a velocity or acceleration of a point in a reference frame is the name of the point, whereas the left superscript identifies the reference frame. Thus, for example, the aforementioned velocity of  $A^*$  in  $N$  is written  ${}^N\mathbf{v}^{A^*}$ , and  ${}^A\mathbf{v}^{B^*}$  represents the velocity of  $B^*$  in  $A$ . Similar conventions are established in connection with angular momenta, kinetic energies, and so forth.

While there are distinct differences between our approach to dynamics, on the one hand, and traditional approaches, on the other hand, there is no fundamental conflict between the new and the old. On the contrary, the material in this book is entirely compatible with the classical literature. Thus, it is the purpose of this book not only to equip students with the skills they need to deal effectively with present-day dynamics problems, but also to bring them into position to interact smoothly with those trained more conventionally.

*Thomas R. Kane*  
*David A. Levinson*



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## TO THE READER

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Each of the seven chapters of this book is divided into sections. A section is identified by two numbers separated by a decimal point, the first number referring to the chapter in which the section appears, and the second identifying the section within the chapter. Thus, the identifier 2.14 refers to the fourteenth section of the second chapter. A section identifier appears at the *top of each page*.

Equations are numbered serially within sections. For example, the equations in Secs. 2.14 and 2.15 are numbered (1)–(31) and (1)–(50), respectively. References to an equation may be made both within the section in which the equation appears and in other sections. In the first case, the equation number is cited as a single number; in the second case, the section number is included as part of a three-number designation. Thus, within Sec. 2.14, Eq. (2) of Sec. 2.14 is referred to as Eq. (2); in Sec. 2.15, the same equation is referred to as Eq. (2.14.2). To locate an equation cited in this manner, one may make use of the section identifiers appearing at the tops of pages.

Figures appearing in the chapters are numbered so as to identify the sections in which the figures appear. For example, the two figures in Sec. 4.8 are designated Fig. 4.8.1 and Fig. 4.8.2. To avoid confusing these figures with those in the problem sets and in Appendix I, the figure number is preceded by the letter P in the case of problem set figures, and by the letter A in the case of Appendix I figures. The double number following the letter P refers to the problem statement in which the figure is introduced. For example, Fig. P12.3 is introduced in Problem 12.3. Similarly, Table 3.4.1 is the designation for a table in Sec. 3.4, and Table P14.6.2 is associated with Problem 14.6.

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*David A. Levinson*



## DIFFERENTIATION OF VECTORS

The discipline of dynamics deals with changes of various kinds, such as changes in the position of a particle in a reference frame, changes in the configuration of a mechanical system, and so forth. To characterize the manner in which some of these changes take place, one employs the differential calculus of vectors, a subject that can be regarded as an extension of material usually taught under the heading of the differential calculus of scalar functions. The extension consists primarily of provisions made to accommodate the fact that *reference frames* play a central role in connection with many of the vectors of interest in dynamics. For example, let  $A$  and  $B$  be reference frames moving relative to each other, but having one point  $O$  in common at all times, and let  $P$  be a point fixed in  $A$ , and thus moving in  $B$ . Then the velocity of  $P$  in  $A$  is equal to zero, whereas the velocity of  $P$  in  $B$  differs from zero. Now, each of these velocities is a time-derivative of the same vector,  $\mathbf{r}^{OP}$ , the position vector from  $O$  to  $P$ . Hence, it is meaningless to speak simply of *the* time-derivative of  $\mathbf{r}^{OP}$ . Clearly, therefore, the calculus used to differentiate vectors must permit one to distinguish between differentiation with respect to a scalar variable in a reference frame  $A$  and differentiation with respect to the same variable in a reference frame  $B$ .

When working with elementary principles of dynamics, such as Newton's second law or the angular momentum principle, one needs only the ordinary differential calculus of vectors, that is, a theory involving differentiations of vectors with respect to a single scalar variable, generally the time. Consideration of advanced principles of dynamics, such as those presented in later chapters of this



book, necessitates, in addition, *partial* differentiation of vectors with respect to several scalar variables, such as generalized coordinates and generalized speeds. Accordingly, the present chapter is devoted to the exposition of definitions, and consequences of these definitions, needed in the chapters that follow.

## 1.1 VECTOR FUNCTIONS

When either the magnitude of a vector  $\mathbf{v}$  and/or the direction of  $\mathbf{v}$  in a reference frame  $A$  depends on a scalar variable  $q$ ,  $\mathbf{v}$  is called a *vector function of  $q$  in  $A$* . Otherwise,  $\mathbf{v}$  is said to be *independent of  $q$  in  $A$* .

**Example** In Fig. 1.1.1,  $P$  represents a point moving on the surface of a rigid sphere  $S$ , which, like any rigid body, may be regarded as a reference frame. (Reference frames should not be confused with coordinate systems. Many coordinate systems can be embedded in a given reference frame.) If  $\mathbf{p}$  is the position vector from the center  $C$  of  $S$  to point  $P$ , and if  $q_1$  and  $q_2$  are the angles shown, then  $\mathbf{p}$  is a vector function of  $q_1$  and  $q_2$  in  $S$  because the direction of  $\mathbf{p}$  in  $S$  depends on  $q_1$  and  $q_2$ , but  $\mathbf{p}$  is independent of  $q_3$  in  $S$ , where  $q_3$  is the distance from  $C$  to a point  $R$  situated as shown in Fig. 1.1.1. The position vector  $\mathbf{r}$  from  $C$  to  $R$  is a vector function of  $q_3$  in  $S$ , but is independent of  $q_1$  and  $q_2$  in  $S$ , and the position vector  $\mathbf{q}$  from  $P$  to  $R$  is a vector function of  $q_1$ ,  $q_2$ , and  $q_3$  in  $S$ .

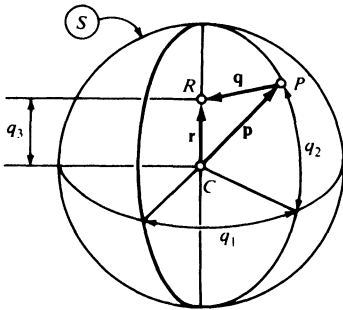


Figure 1.1.1

## 1.2 SEVERAL REFERENCE FRAMES

A vector  $\mathbf{v}$  may be a function of a variable  $q$  in one reference frame, but be independent of  $q$  in another reference frame.

**Example** The outer gimbal ring  $A$ , inner gimbal ring  $B$ , and rotor  $C$  of the gyroscope depicted in Fig. 1.2.1 each can be regarded as a reference frame. If  $\mathbf{p}$  is the position vector from point  $O$  to a point  $P$  of  $C$ , then  $\mathbf{p}$  is a function of



Conversely, if  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{a}_3$  are mutually perpendicular unit vectors and Eqs. (2) are regarded as definitions of  $v_i$  ( $i = 1, 2, 3$ ), then it follows from Eq. (3) that  $\mathbf{v}$  can be expressed as in Eq. (1).

**Example** In Fig. 1.3.1, which shows the gyroscope considered in the example in Sec. 1.2,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  and  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  designate mutually perpendicular unit vectors fixed in  $A$  and in  $B$ , respectively. The vector  $\mathbf{p}$  can be expressed both as

$$\mathbf{p} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 \quad (4)$$

and as

$$\mathbf{p} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 \quad (5)$$

where  $\alpha_i$  and  $\beta_i$  ( $i = 1, 2, 3$ ) are functions of  $q_1, q_2$ , and  $q_3$ . To determine these functions, note that, if  $C$  has a radius  $R$ , one can proceed from  $O$  to  $P$  by moving through the distances  $R \cos q_1$  and  $R \sin q_1$  in the directions of  $\mathbf{b}_2$  and  $\mathbf{b}_3$  (see Fig. 1.3.2), respectively, which means that

$$\mathbf{p} = R(c_1 \mathbf{b}_2 + s_1 \mathbf{b}_3) \quad (6)$$

where  $c_1$  and  $s_1$  are abbreviations for  $\cos q_1$  and  $\sin q_1$ , respectively. Comparing Eqs. (5) and (6), one thus finds that

$$\beta_1 = 0 \quad \beta_2 = Rc_1 \quad \beta_3 = Rs_1 \quad (7)$$

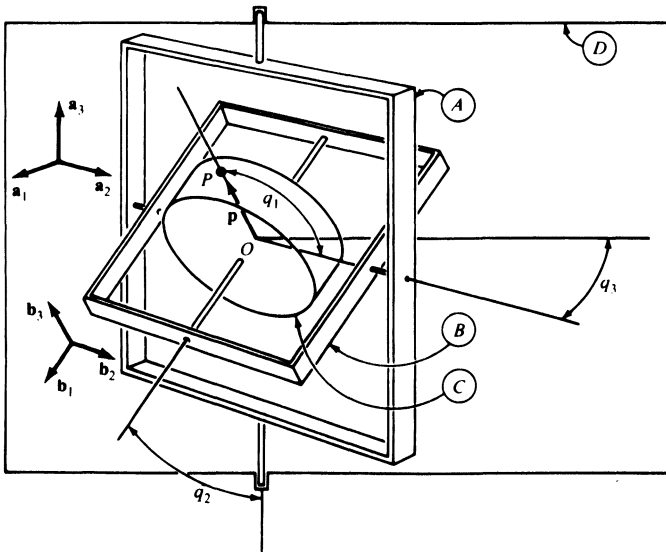
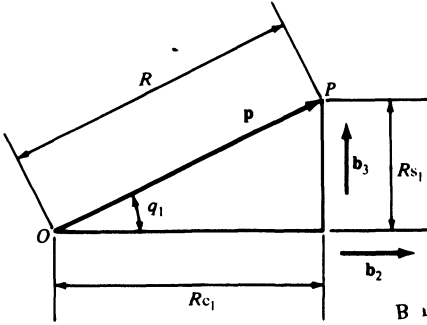


Figure 1.3.1



B L Figure 1.3.2

Moreover, in view of Eq. (4), one can write†

$$\alpha_1 = \underset{(2)}{\mathbf{p}} \cdot \mathbf{a}_1 = \underset{(6)}{R(c_1 \mathbf{b}_2 \cdot \mathbf{a}_1 + s_1 \mathbf{b}_3 \cdot \mathbf{a}_1)} \quad (8)$$

$$\alpha_2 = \underset{(2)}{\mathbf{p}} \cdot \mathbf{a}_2 = \underset{(6)}{R(c_1 \mathbf{b}_2 \cdot \mathbf{a}_2 + s_1 \mathbf{b}_3 \cdot \mathbf{a}_2)} \quad (9)$$

$$\alpha_3 = \underset{(2)}{\mathbf{p}} \cdot \mathbf{a}_3 = \underset{(6)}{R(c_1 \mathbf{b}_2 \cdot \mathbf{a}_3 + s_1 \mathbf{b}_3 \cdot \mathbf{a}_3)} \quad (10)$$

From Fig. 1.3.1,

$$\mathbf{b}_2 \cdot \mathbf{a}_1 = 0 \quad \mathbf{b}_2 \cdot \mathbf{a}_2 = 1 \quad \mathbf{b}_2 \cdot \mathbf{a}_3 = 0 \quad (11)$$

$$\mathbf{b}_3 \cdot \mathbf{a}_1 = c_2 \quad \mathbf{b}_3 \cdot \mathbf{a}_2 = 0 \quad \mathbf{b}_3 \cdot \mathbf{a}_3 = s_2 \quad (12)$$

Hence, the  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  measure numbers of  $\mathbf{p}$  are

$$\alpha_1 = \underset{(8)}{R s_1 c_2} \quad \alpha_2 = \underset{(9)}{R c_1} \quad \alpha_3 = \underset{(10)}{R s_1 s_2} \quad (13)$$

respectively.

## 1.4 FIRST DERIVATIVES

If  $\mathbf{v}$  is a vector function of  $n$  scalar variables  $q_1, \dots, q_n$  in a reference frame  $A$  (see Sec. 1.1), then  $n$  vectors, called *first partial derivatives of  $\mathbf{v}$  in  $A$*  and denoted by the symbols

$$\frac{{}^A \partial \mathbf{v}}{\partial q_r} \quad \text{or} \quad \frac{{}^A \partial}{\partial q_r} (\mathbf{v}) \quad \text{or} \quad {}^A \partial \mathbf{v} / \partial q_r \quad (r = 1, \dots, n)$$

† Numbers beneath signs of equality or beneath other symbols refer to equations numbered correspondingly.

are defined as follows: Let  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  be any nonparallel, noncoplanar unit vectors fixed in  $A$ , and let  $v_i$  be the  $\mathbf{a}_i$  measure number of  $\mathbf{v}$  (see Sec. 1.3). Then

$$\frac{{}^A\partial\mathbf{v}}{\partial q_r} \triangleq \sum_{i=1}^3 \frac{\partial v_i}{\partial q_r} \mathbf{a}_i \quad (r = 1, \dots, n) \quad (1)$$

When  $\mathbf{v}$  is regarded as a vector function of only a single scalar variable in  $A$ —for instance the time  $t$ —then this definition reduces to that of the *ordinary derivative of  $\mathbf{v}$  with respect to  $t$  in  $A$* , that is, to†

$$\frac{{}^A d\mathbf{v}}{dt} \triangleq \sum_{i=1}^3 \frac{dv_i}{dt} \mathbf{a}_i \quad (2)$$

**Example** The vector  $\mathbf{p}$  considered in the example in Sec. 1.3 possesses partial derivatives with respect to  $q_1, q_2$ , and  $q_3$  in each of the reference frames  $A, B$ , and  $C$ . To form  ${}^A\partial\mathbf{p}/\partial q_r$  ( $r = 1, 2, 3$ ), one can use the  $\mathbf{a}_i$  ( $i = 1, 2, 3$ ) measure numbers of  $\mathbf{p}$  available in Eqs. (1.3.13) and thus write

$$\frac{{}^A\partial\mathbf{p}}{\partial q_r} = \left[ \frac{\partial}{\partial q_r} (Rs_1c_2) \right] \mathbf{a}_1 + \left[ \frac{\partial}{\partial q_r} (Rc_1) \right] \mathbf{a}_2 + \left[ \frac{\partial}{\partial q_r} (Rs_1s_2) \right] \mathbf{a}_3 \quad (r = 1, 2, 3) \quad (3)$$

Consequently,

$$\frac{{}^A\partial\mathbf{p}}{\partial q_1} = R(c_1c_2\mathbf{a}_1 - s_1\mathbf{a}_2 + c_1s_2\mathbf{a}_3) \quad (4)$$

$$\frac{{}^A\partial\mathbf{p}}{\partial q_2} = Rs_1(-s_2\mathbf{a}_1 + c_2\mathbf{a}_3) \quad (5)$$

$$\frac{{}^A\partial\mathbf{p}}{\partial q_3} = 0 \quad (6)$$

The last result agrees with the statement in the example in Sec. 1.2 that  $\mathbf{p}$  is independent of  $q_3$  in  $A$ .

Proceeding similarly to determine  ${}^B\partial\mathbf{p}/\partial q_r$  ( $r = 1, 2, 3$ ), one obtains with the aid of Eqs. (1.3.7),

$$\frac{{}^B\partial\mathbf{p}}{\partial q_1} = R(-s_1\mathbf{b}_2 + c_1\mathbf{b}_3) \quad \frac{{}^B\partial\mathbf{p}}{\partial q_2} = 0 \quad \frac{{}^B\partial\mathbf{p}}{\partial q_3} = 0 \quad (7)$$

Finally, since  $\mathbf{p}$  is independent of  $q_r$  ( $r = 1, 2, 3$ ) in  $C$ ,

$$\frac{{}^C\partial\mathbf{p}}{\partial q_r} = 0 \quad (r = 1, 2, 3) \quad (8)$$

† The importance of reference frames in connection with time-differentiation of vectors is obscured by defining the ordinary derivative of a vector  $\mathbf{v}$  with respect to time  $t$  as the limit of  $\Delta\mathbf{v}/\Delta t$  as  $\Delta t$  approaches zero, for this fails to bring any reference frame into evidence.

Suppose now that  $q_1$  and  $q_2$  are specified as explicit functions of time  $t$ , namely,

$$q_1 = t \quad q_2 = 2t \quad q_3 = 3t \quad (9)$$

Then  $\alpha_i$  ( $i = 1, 2, 3$ ) of Eq. (1.3.4) can be expressed as [see Eqs. (1.3.13)]

$$\alpha_1 = R \sin t \cos 2t \quad \alpha_2 = R \cos t \quad \alpha_3 = R \sin t \sin 2t \quad (10)$$

and the ordinary derivative of  $\mathbf{p}$  with respect to  $t$  in  $A$  is seen to be given by

$$\begin{aligned} \frac{{}^A d\mathbf{p}}{dt} &= \frac{d\alpha_1}{dt} \mathbf{a}_1 + \frac{d\alpha_2}{dt} \mathbf{a}_2 + \frac{d\alpha_3}{dt} \mathbf{a}_3 \\ &= R[(\cos t \cos 2t - 2 \sin t \sin 2t) \mathbf{a}_1 - \sin t \mathbf{a}_2 \\ &\quad + (\cos t \sin 2t + 2 \sin t \cos 2t) \mathbf{a}_3] \end{aligned} \quad (11)$$

while the ordinary derivative of  $\mathbf{p}$  with respect to  $t$  in  $B$  is [see Eqs. (1.3.7)]

$$\frac{{}^B d\mathbf{p}}{dt} = R(-\sin t \mathbf{b}_2 + \cos t \mathbf{b}_3) \quad (12)$$

Finally,

$$\frac{{}^C d\mathbf{p}}{dt} = 0 \quad (13)$$

because, when  $\mathbf{p}$  is expressed as

$$\mathbf{p} = \gamma_1 \mathbf{c}_1 + \gamma_2 \mathbf{c}_2 + \gamma_3 \mathbf{c}_3 \quad (14)$$

where  $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$  are unit vectors fixed in  $C$ , then  $\gamma_1, \gamma_2, \gamma_3$  are necessarily constants,  $\mathbf{p}$  being fixed in  $C$ .

## 1.5 REPRESENTATIONS OF DERIVATIVES

When a partial or ordinary derivative of a vector  $\mathbf{v}$  in a reference frame  $A$  is formed by carrying out the operations indicated in Eqs. (1.4.1) and (1.4.2), the resulting expression involves the unit vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , that is, unit vectors fixed in  $A$ . By expressing each of these unit vectors in terms of unit vectors  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  fixed in a reference frame  $B$ , one arrives at a new representation of the derivative under consideration, namely, one involving  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ , but one is still dealing with derivatives of  $\mathbf{v}$  in  $A$ , not in  $B$ , unless these two derivatives happen to be equal to each other.

**Example** Referring to Eq. (1.4.4) and noting (see Fig. 1.3.1) that

$$\mathbf{a}_1 = s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3 \quad \mathbf{a}_2 = \mathbf{b}_2 \quad \mathbf{a}_3 = -c_2 \mathbf{b}_1 + s_2 \mathbf{b}_3 \quad (1)$$

one can write

$$\begin{aligned}\frac{{}^A\partial\mathbf{p}}{\partial q_1} &= R[c_1c_2(s_2\mathbf{b}_1 + c_2\mathbf{b}_3) - s_1\mathbf{b}_2 + c_1s_2(-c_2\mathbf{b}_1 + s_2\mathbf{b}_3)] \\ &= R(-s_1\mathbf{b}_2 + c_1\mathbf{b}_3)\end{aligned}\quad (2)$$

The right-hand sides of this equation and of the first of Eqs. (1.4.7) are identical. Hence, it appears that we have produced  ${}^B\partial\mathbf{p}/\partial q_1$  by expressing  ${}^A\partial\mathbf{p}/\partial q_1$  in terms of  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ . It was possible to do this because  ${}^A\partial\mathbf{p}/\partial q_1$  and  ${}^B\partial\mathbf{p}/\partial q_1$  happen to be equal to each other. To see that the same procedure does not lead to  ${}^B\partial\mathbf{p}/\partial q_2$  when one starts with  ${}^A\partial\mathbf{p}/\partial q_2$ , refer to Eq. (1.4.5) to write, with the aid of Eqs. (1),

$$\begin{aligned}\frac{{}^A\partial\mathbf{p}}{\partial q_2} &= Rs_1[-s_2(s_2\mathbf{b}_1 + c_2\mathbf{b}_3) + c_2(-c_2\mathbf{b}_1 + s_2\mathbf{b}_3)] \\ &= -Rs_1\mathbf{b}_1\end{aligned}\quad (3)$$

and compare this with the second of Eqs. (1.4.7).

## 1.6 NOTATION FOR DERIVATIVES

In general, both partial and ordinary derivatives of a vector  $\mathbf{v}$  in a reference frame  $A$  differ from corresponding derivatives in any other reference frame  $B$ . It follows that notations such as  $\partial\mathbf{v}/\partial q_r$  or  $d\mathbf{v}/dt$ —that is, ones that involve no mention of any reference frame—are meaningful only either when the context in which they appear clearly implies a particular reference frame or when it does not matter which reference frame is used. In the sequel, it is to be understood that, whenever no reference frame is mentioned explicitly, any reference frame may be used, but all partial or ordinary differentiations indicated in any one equation are meant to be performed in the same reference frame.

**Example** The equations

$$\mathbf{p} \cdot \frac{\partial\mathbf{p}}{\partial q_r} = 0 \quad (r = 1, 2, 3) \quad (1)$$

and

$$\mathbf{p} \cdot \frac{d\mathbf{p}}{dt} = 0 \quad (2)$$

are valid for the vector  $\mathbf{p}$  and the quantities  $q_1, q_2, q_3$  introduced in the example in Sec. 1.2, regardless of the reference frame in which  $\mathbf{p}$  is differentiated. We shall shortly be in a position to prove this. To verify it for a few specific cases, refer to Eqs. (1.3.6) and (1.4.7), which yield

$$\mathbf{p} \cdot \frac{{}^B\partial\mathbf{p}}{\partial q_1} = R^2(c_1\mathbf{b}_2 + s_1\mathbf{b}_3) \cdot (-s_1\mathbf{b}_2 + c_1\mathbf{b}_3) = 0 \quad (3)$$



or use Eqs. (1.3.4), (1.3.13), and (1.4.5) to write

$$\mathbf{p} \cdot \frac{\partial \mathbf{p}}{\partial q_2} = R^2 s_1 (s_1 c_2 \mathbf{a}_1 + c_1 \mathbf{a}_2 + s_1 s_2 \mathbf{a}_3) \cdot (-s_2 \mathbf{a}_1 + c_2 \mathbf{a}_3) = 0 \quad (4)$$

Finally, note that Eqs. (1.3.4), (1.4.10), (1.4.11), and (1.4.13) lead to

$$\mathbf{p} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{p} \cdot \frac{c d\mathbf{p}}{dt} = 0 \quad (5)$$

## 1.7 DIFFERENTIATION OF SUMS AND PRODUCTS

As an immediate consequence of the definition given in Eqs. (1.4.1), the following rules govern the differentiation of sums and products involving vector functions.

If  $\mathbf{v}_1, \dots, \mathbf{v}_N$  are vector functions of the scalar variables  $q_1, \dots, q_n$  in some reference frame, then

$$\frac{\partial}{\partial q_r} \sum_{i=1}^N \mathbf{v}_i = \sum_{i=1}^N \frac{\partial \mathbf{v}_i}{\partial q_r} \quad (r = 1, \dots, n) \quad (1)$$

If  $s$  is a scalar function of  $q_1, \dots, q_n$ , and  $\mathbf{v}$  and  $\mathbf{w}$  are vector functions of these variables in some reference frame, then

$$\frac{\partial}{\partial q_r} (s\mathbf{v}) = \frac{\partial s}{\partial q_r} \mathbf{v} + s \frac{\partial \mathbf{v}}{\partial q_r} \quad (r = 1, \dots, n) \quad (2)$$

$$\frac{\partial}{\partial q_r} (\mathbf{v} \cdot \mathbf{w}) = \frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{w} + \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial q_r} \quad (r = 1, \dots, n) \quad (3)$$

$$\frac{\partial}{\partial q_r} (\mathbf{v} \times \mathbf{w}) = \frac{\partial \mathbf{v}}{\partial q_r} \times \mathbf{w} + \mathbf{v} \times \frac{\partial \mathbf{w}}{\partial q_r} \quad (r = 1, \dots, n) \quad (4)$$

More generally, if  $P$  is the product of  $N$  scalar and/or vector functions  $F_i$  ( $i = 1, \dots, N$ ), that is, if

$$P = F_1 F_2 \cdots F_N \quad (5)$$

then, if all symbols of operation, such as dots, crosses, and parentheses, are kept in place,

$$\frac{\partial P}{\partial q_r} = \frac{\partial F_1}{\partial q_r} F_2 \cdots F_N + F_1 \frac{\partial F_2}{\partial q_r} \cdots F_N + \cdots + F_1 F_2 \cdots F_{N-1} \frac{\partial F_N}{\partial q_r} \quad (r = 1, \dots, n) \quad (6)$$

Relationships analogous to Eqs. (1)–(6) govern the ordinary differentiation [see Eq. (1.4.2)] of vector and/or scalar functions of a single scalar variable.

**Example** By definition, the square of a vector  $\mathbf{v}$ , written  $\mathbf{v}^2$ , is the scalar quantity obtained by dot-multiplying  $\mathbf{v}$  with  $\mathbf{v}$ . Hence, if  $s$  is a scalar function of  $q_1, \dots, q_n$ , then

$$\begin{aligned} \frac{\partial}{\partial q_r} (s\mathbf{v}^2) &= \frac{\partial}{\partial q_r} (s\mathbf{v} \cdot \mathbf{v}) \\ &= \frac{\partial s}{\partial q_r} \mathbf{v} \cdot \mathbf{v} + s \frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{v} + s\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_r} \quad (r = 1, \dots, n) \end{aligned} \quad (7)$$

or, since the last two terms are equal to each other,

$$\frac{\partial}{\partial q_r} (s\mathbf{v}^2) = \frac{\partial s}{\partial q_r} \mathbf{v}^2 + 2s\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_r} \quad (r = 1, \dots, n) \quad (8)$$

This result can be used to establish the validity of Eqs. (1.6.1) by taking  $s = 1$ , writing  $\mathbf{p}$  in place of  $\mathbf{v}$ , and letting  $n = 3$ , which yields

$$\frac{\partial}{\partial q_r} (\mathbf{p}^2) \underset{(8)}{=} 2\mathbf{p} \cdot \frac{\partial \mathbf{p}}{\partial q_r} \quad (r = 1, 2, 3) \quad (9)$$

and then noting that, for the vector  $\mathbf{p}$  in Eqs. (1.6.1),  $\mathbf{p}^2$  is a constant, so that

$$\frac{\partial}{\partial q_r} (\mathbf{p}^2) = 0 \quad (r = 1, 2, 3) \quad (10)$$

Since  $2 \neq 0$ , Eqs. (9) and (10) imply that

$$\mathbf{p} \cdot \frac{\partial \mathbf{p}}{\partial q_r} = 0 \quad (r = 1, 2, 3) \quad (11)$$

in agreement with Eqs. (1.6.1).

## 1.8 SECOND DERIVATIVES

In general,  ${}^A\partial\mathbf{v}/\partial q_r$  (see Sec. 1.4) is a vector function of  $q_1, \dots, q_n$  both in  $A$  and in any other reference frame  $B$  and can, therefore, be differentiated with respect to any one of  $q_1, \dots, q_n$  both in  $A$  and in  $B$ . The result of such a differentiation is called a *second partial derivative*. Similarly, the ordinary derivative  ${}^A d\mathbf{v}/dt$  (see Sec. 1.4) can be differentiated with respect to  $t$  both in  $A$  and in any other reference frame  $B$ .

The order in which successive differentiations are performed can affect the results. For example, in general,

$$\frac{{}^B\partial}{\partial q_s} \left( \frac{{}^A\partial \mathbf{v}}{\partial q_r} \right) \neq \frac{{}^A\partial}{\partial q_r} \left( \frac{{}^B\partial \mathbf{v}}{\partial q_s} \right) \quad (r, s = 1, \dots, n) \quad (1)$$

and

$$\frac{{}^B d}{dt} \left( \frac{{}^A d\mathbf{v}}{dt} \right) \neq \frac{{}^A d}{dt} \left( \frac{{}^B d\mathbf{v}}{dt} \right) \quad (2)$$

However, if successive partial differentiations with respect to various variables are performed in the same reference frame, then the order is immaterial; that is,

$$\frac{\partial}{\partial q_s} \left( \frac{\partial \mathbf{v}}{\partial q_r} \right) = \frac{\partial}{\partial q_r} \left( \frac{\partial \mathbf{v}}{\partial q_s} \right) \quad (r, s = 1, \dots, n) \quad (3)$$

and

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathbf{v}}{\partial q_r} \right) = \frac{\partial}{\partial q_r} \left( \frac{\partial \mathbf{v}}{\partial t} \right) \quad (r = 1, \dots, n) \quad (4)$$

**Example** Referring to the example in Sec. 1.3, suppose that a vector  $\mathbf{v}$  is given by

$$\mathbf{v} = t \mathbf{a}_1 \quad (5)$$

and that  $q_2$  is a specified function of  $t$ . Then

$$\frac{{}^A d \mathbf{v}}{dt} = \mathbf{a}_1 = s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3 \quad (5) \quad (1.5.1) \quad (6)$$

and

$$\frac{{}^B d}{dt} \left( \frac{{}^A d \mathbf{v}}{dt} \right) = \dot{q}_2 (c_2 \mathbf{b}_1 - s_2 \mathbf{b}_3) = -\dot{q}_2 \mathbf{a}_3 \quad (6) \quad (1.5.1) \quad (7)$$

Also,

$$\mathbf{v} = t(s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3) \quad (5) \quad (1.5.1) \quad (8)$$

so that

$$\begin{aligned} \frac{{}^B d \mathbf{v}}{dt} &= \frac{dt}{dt} (s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3) + t \frac{{}^B d}{dt} (s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3) \\ &= s_2 \mathbf{b}_1 + c_2 \mathbf{b}_3 + t \dot{q}_2 (c_2 \mathbf{b}_1 - s_2 \mathbf{b}_3) \\ &= \mathbf{a}_1 - t \dot{q}_2 \mathbf{a}_3 \quad (1.5.1) \end{aligned} \quad (9)$$

and

$$\frac{{}^A d}{dt} \left( \frac{{}^B d \mathbf{v}}{dt} \right) = -(\dot{q}_2 + t \ddot{q}_2) \mathbf{a}_3 \quad (9) \quad (10)$$

Comparing Eqs. (7) and (10), one sees that, in general, one must expect the result of successive differentiations in various reference frames to depend on the order in which the differentiations are performed.

## 1.9 TOTAL AND PARTIAL DERIVATIVES

If  $q_1, \dots, q_n$  are scalar functions of a single variable  $t$ , it is sometimes convenient to regard a vector  $\mathbf{v}$  as a vector function of the  $n + 1$  independent variables  $q_1, \dots, q_n$ , and  $t$  in a reference frame  $A$ . The ordinary derivative of  $\mathbf{v}$  with respect to  $t$  in  $A$  (see Sec. 1.4), called a *total derivative* under these circumstances, then can be expressed in terms of partial derivatives as

$$\frac{{}^A d\mathbf{v}}{dt} = \sum_{r=1}^n \frac{{}^A \partial \mathbf{v}}{\partial q_r} \dot{q}_r + \frac{{}^A \partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}}{\partial q_1} \frac{\partial q_1}{\partial t} + \dots + \frac{\partial \mathbf{v}}{\partial t} \frac{\partial t}{\partial t} \quad (1)$$

where  $\dot{q}_r$  denotes the first derivative of  $q_r$  with respect to  $t$ . Moreover, if  $\mathbf{v}$  is differentiated both totally with respect to  $t$  and partially with respect to  $q_r$ , then the order in which the differentiations are performed is immaterial; that is,

$$\frac{d}{dt} \frac{\partial \mathbf{v}}{\partial q_r} = \frac{\partial}{\partial q_r} \frac{d\mathbf{v}}{dt} \quad (r = 1, \dots, n) \quad (2)$$

**Derivations** Let  $\mathbf{a}_i$  ( $i = 1, 2, 3$ ) be nonparallel, noncoplanar unit vectors fixed in  $A$ , and regard  $v_i$ , the  $\mathbf{a}_i$  measure number of  $\mathbf{v}$  (see Sec. 1.3), as a function of  $q_1, \dots, q_n$ , and  $t$ . From scalar calculus,

$$\frac{dv_i}{dt} = \sum_{r=1}^n \frac{\partial v_i}{\partial q_r} \dot{q}_r + \frac{\partial v_i}{\partial t} \quad (i = 1, 2, 3) \quad (3)$$

and, if  $m$  and  $q_m$  are defined as

$$m \triangleq n + 1 \quad (4)$$

and

$$q_m \triangleq t \quad (5)$$

so that

$$\dot{q}_m \underset{(5)}{=} 1 \quad (6)$$

then Eqs. (3) can be rewritten as

$$\frac{dv_i}{dt} = \sum_{r=1}^n \frac{\partial v_i}{\partial q_r} \dot{q}_r + \underset{(5)}{\frac{\partial v_i}{\partial q_m}} \underset{(6)}{\dot{q}_m} = \sum_{r=1}^m \frac{\partial v_i}{\partial q_r} \dot{q}_r \quad (i = 1, 2, 3) \quad (7)$$

and substitution into Eq. (1.4.2) yields

$$\begin{aligned} \frac{{}^A d\mathbf{v}}{dt} &= \sum_{i=1}^3 \left( \sum_{r=1}^m \frac{\partial v_i}{\partial q_r} \dot{q}_r \right) \mathbf{a}_i = \sum_{r=1}^m \left( \sum_{i=1}^3 \frac{\partial v_i}{\partial q_r} \mathbf{a}_i \right) \dot{q}_r \\ &= \sum_{r=1}^m \underset{(1.4.1)}{{}^A \frac{\partial \mathbf{v}}{\partial q_r}} \dot{q}_r = \sum_{r=1}^n \frac{{}^A \partial \mathbf{v}}{\partial q_r} \dot{q}_r + \frac{{}^A \partial \mathbf{v}}{\partial q_m} \dot{q}_m \end{aligned} \quad (8)$$

which, in view of Eqs. (5) and (6), establishes the validity of Eq. (1). Furthermore, replacing  $\mathbf{v}$  in Eq. (1) with  $\partial \mathbf{v} / \partial q_s$  produces

$$\begin{aligned}
 \frac{d}{dt} \frac{\partial \mathbf{v}}{\partial q_s} &= \sum_{r=1}^n \left[ \frac{\partial}{\partial q_r} \left( \frac{\partial \mathbf{v}}{\partial q_s} \right) \right] \dot{q}_r + \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{v}}{\partial q_s} \right) \\
 &= \sum_{r=1}^n \left[ \frac{\partial}{\partial q_s} \left( \frac{\partial \mathbf{v}}{\partial q_r} \right) \right] \dot{q}_r + \frac{\partial}{\partial q_s} \left( \frac{\partial \mathbf{v}}{\partial t} \right) \\
 &\quad (1.8.3) \qquad (1.8.4) \\
 &\stackrel{(1.7.1)}{=} \frac{\partial}{\partial q_s} \left( \sum_{r=1}^n \frac{\partial \mathbf{v}}{\partial q_r} \dot{q}_r + \frac{\partial \mathbf{v}}{\partial t} \right) \\
 &\stackrel{(1)}{=} \frac{\partial}{\partial q_s} \frac{d\mathbf{v}}{dt} \quad (s = 1, \dots, n) \tag{9}
 \end{aligned}$$

in agreement with Eq. (2).

**Example** To see that one can proceed in a variety of ways to find the ordinary derivative of a vector function in a reference frame, consider once more the vector  $\mathbf{p}$  introduced in the example in Sec. 1.3, and again let  $q_1$ ,  $q_2$ , and  $q_3$  be given by Eqs. (1.4.9). Then the ordinary time-derivative of  $\mathbf{p}$  in  $A$ , previously found by using Eq. (1.4.2), is given by Eq. (1.4.11). Now refer to Eqs. (1.3.4) and (1.3.13) to express  $\mathbf{p}$  as

$$\mathbf{p} = R(s_1 c_2 \mathbf{a}_1 + c_1 \mathbf{a}_2 + s_1 s_2 \mathbf{a}_3) \tag{10}$$

and use Eqs. (1.4.9) to rewrite the  $\mathbf{a}_3$  measure number of  $\mathbf{p}$  as an explicit function of  $t$ , that is, to replace Eq. (10) with

$$\mathbf{p} = R(s_1 c_2 \mathbf{a}_1 + c_1 \mathbf{a}_2 + \sin t \sin 2t \mathbf{a}_3) \tag{11}$$

Furthermore, regard  $\mathbf{p}$  as a function of the independent variables  $q_1$ ,  $q_2$ ,  $q_3$ , and  $t$ , and then appeal to Eq. (1) to write

$$\begin{aligned}
 \frac{d\mathbf{p}}{dt} &= \frac{\partial \mathbf{p}}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{p}}{\partial q_2} \dot{q}_2 + \frac{\partial \mathbf{p}}{\partial q_3} \dot{q}_3 + \frac{\partial \mathbf{p}}{\partial t} \\
 &\stackrel{(1.1)}{=} R[(c_1 c_2 \mathbf{a}_1 - s_1 \mathbf{a}_2) \dot{q}_1 + (-s_1 s_2 \mathbf{a}_1) \dot{q}_2 \\
 &\quad + (\cos t \sin 2t + 2 \sin t \cos 2t) \mathbf{a}_3] \tag{12}
 \end{aligned}$$

Finally, make the substitutions [see Eqs. (1.4.9)]

$$\dot{q}_1 = 1 \quad s_1 = \sin t \quad c_1 = \cos t \tag{13}$$

$$\dot{q}_2 = 2 \quad s_2 = \sin 2t \quad c_2 = \cos 2t \tag{14}$$

and verify that the resulting equation is precisely Eq. (1.4.11). The point here is not that use of Eq. (1) facilitates the evaluation of ordinary derivatives; indeed, it may complicate matters. What is important is to realize that one may

treat the same vector in a variety of ways, that the formalism one uses to construct the ordinary derivative of the vector in a given reference frame depends on the functional character one attributes to the vector, but that the result one obtains is independent of the approach taken. In the sequel, Eq. (1) will be used primarily in the course of certain derivations, rather than for the actual evaluation of ordinary derivatives.

## KINEMATICS

Considerations of kinematics play a central role in dynamics. Indeed, one's effectiveness in formulating equations of motion depends primarily on one's ability to construct correct mathematical expressions for kinematical quantities such as angular velocities of rigid bodies, velocities of points, and so forth. Therefore, mastery of the material in this chapter is essential.

The sections that follow can be divided into four groups. Sections 2.1–2.5, which form the first group, are concerned with *rotational motion of a rigid body*. The principal kinematical quantity introduced here is the angular velocity of a rigid body in a reference frame. Next, *translational motion of a point* is treated in Secs. 2.6–2.8, where four theorems frequently used in practice are derived from definitions of the velocity and acceleration of a point in a reference frame. (The reason for discussing translational motion after rotational motion is that the theorems on translational motion in Secs. 2.6–2.8 involve angular velocities and angular accelerations of rigid bodies, whereas the material on rotational motion in Secs. 2.1–2.5 does not involve velocities or accelerations of points.) Thereafter, in Secs. 2.9–2.13, the subject of *constraints* is examined in detail, and mathematical techniques for dealing with constraints are presented in terms involving *generalized coordinates* and *generalized speeds*. Finally, *partial angular velocities* of a rigid body and *partial velocities* of a point are at the focus of attention in Secs. 2.14 and 2.15. It is these quantities that ultimately enable one to form in a straightforward manner the terms that make up dynamical equations of motion.

### 2.1 ANGULAR VELOCITY

The use of angular velocities greatly facilitates the analysis of motions of systems containing rigid bodies. We begin our discussion of this topic with a formal

definition of angular velocity; while it is abstract, this definition provides a sound basis for the derivation of theorems [see, for example, Eq. (2)] used to solve physical problems.†

Let  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a right-handed set of mutually perpendicular unit vectors fixed in a rigid body  $B$  moving in a reference frame  $A$ . The *angular velocity* of  $B$  in  $A$ , denoted by  ${}^A\boldsymbol{\omega}^B$ , is defined as

$${}^A\boldsymbol{\omega}^B \triangleq \mathbf{b}_1 \frac{{}^Ad\mathbf{b}_2}{dt} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{{}^Ad\mathbf{b}_3}{dt} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{{}^Ad\mathbf{b}_1}{dt} \cdot \mathbf{b}_2 \quad (1)$$

One task facilitated by the use of angular velocity vectors is the time-differentiation of vectors fixed in a rigid body, for it enables one to obtain the first time-derivative of such a vector by performing a cross-multiplication. Specifically, if  $\boldsymbol{\beta}$  is any vector fixed in  $B$ , then

$$\frac{{}^Ad\boldsymbol{\beta}}{dt} = {}^A\boldsymbol{\omega}^B \times \boldsymbol{\beta} \quad (2)$$

**Derivation** Using dots to denote time-differentiation in  $A$ , one can rewrite Eq. (1) as

$${}^A\boldsymbol{\omega}^B \triangleq \mathbf{b}_1 \dot{\mathbf{b}}_2 \cdot \mathbf{b}_3 + \mathbf{b}_2 \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 + \mathbf{b}_3 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \quad (3)$$

and cross-multiplication of Eq. (3) with  $\mathbf{b}_1$  gives

$${}^A\boldsymbol{\omega}^B \times \mathbf{b}_1 = \underset{(3)}{\mathbf{b}_2 \times \mathbf{b}_1 \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1} + \mathbf{b}_3 \times \mathbf{b}_1 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \quad (4)$$

Now, since  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a right-handed set of mutually perpendicular unit vectors, each can be expressed as a cross-product involving the remaining two. For example,

$$\mathbf{b}_2 = \mathbf{b}_3 \times \mathbf{b}_1 \quad \mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 \quad (5)$$

and substitution into Eq. (4) yields

$${}^A\boldsymbol{\omega}^B \times \mathbf{b}_1 = - \underset{(4)}{\mathbf{b}_3 \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1} + \underset{(5)}{\mathbf{b}_2 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2} \quad (6)$$

Moreover, time-differentiation of the equations  $\mathbf{b}_1 \cdot \mathbf{b}_1 = 1$  and  $\mathbf{b}_3 \cdot \mathbf{b}_1 = 0$  produces

$$\dot{\mathbf{b}}_1 \cdot \mathbf{b}_1 = 0 \quad \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 = -\dot{\mathbf{b}}_1 \cdot \mathbf{b}_3 \quad (7)$$

and with the aid of these one can rewrite Eq. (6) as

$${}^A\boldsymbol{\omega}^B \times \mathbf{b}_1 = \underset{(6)}{\mathbf{b}_1 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_1} + \underset{(7)}{\mathbf{b}_2 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2} + \underset{(7)}{\mathbf{b}_3 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_3} \quad (8)$$

† The frequently employed definition of angular velocity as the limit of  $\Delta\theta/\Delta t$  as  $\Delta t$  approaches zero is deficient in this regard.



But the right-hand member of this equation is simply a way of writing  $\dot{\mathbf{b}}_1$  [see Eq. (1.3.3)]. Consequently,

$${}^A\omega^B \times \mathbf{b}_1 = \dot{\mathbf{b}}_1 \quad (8)$$

Similarly,

$${}^A\omega^B \times \mathbf{b}_2 = \dot{\mathbf{b}}_2 \quad {}^A\omega^B \times \mathbf{b}_3 = \dot{\mathbf{b}}_3 \quad (10)$$

and, after expressing any vector  $\boldsymbol{\beta}$  fixed in  $B$  as

$$\boldsymbol{\beta} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 \quad (11)$$

where  $\beta_1, \beta_2, \beta_3$  are constants, so that

$$\dot{\boldsymbol{\beta}} = \beta_1 \dot{\mathbf{b}}_1 + \beta_2 \dot{\mathbf{b}}_2 + \beta_3 \dot{\mathbf{b}}_3 \quad (12)$$

one arrives at

$$\begin{aligned} \dot{\boldsymbol{\beta}} &= \beta_1 {}^A\omega^B \times \mathbf{b}_1 + \beta_2 {}^A\omega^B \times \mathbf{b}_2 + \beta_3 {}^A\omega^B \times \mathbf{b}_3 \\ &= {}^A\omega^B \times (\beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3) = {}^A\omega^B \times \boldsymbol{\beta} \end{aligned} \quad (13)$$

**Examples** Figure 2.1.1 shows a rigid satellite  $B$  in orbit about the Earth  $A$ . A dextral set of mutually perpendicular unit vectors  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  is fixed in  $B$ , and a similar such set,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , is fixed in  $A$ . Measurements are made to determine the time-histories of  $\alpha_i, \beta_i, \gamma_i$ , defined as

$$\alpha_i \triangleq \mathbf{b}_1 \cdot \mathbf{a}_i \quad \beta_i \triangleq \mathbf{b}_2 \cdot \mathbf{a}_i \quad \gamma_i \triangleq \mathbf{b}_3 \cdot \mathbf{a}_i \quad (i = 1, 2, 3) \quad (14)$$

as well as the time-histories of  $\dot{\alpha}_i, \dot{\beta}_i, \dot{\gamma}_i$ , defined as the time-derivatives of  $\alpha_i, \beta_i, \gamma_i$ , respectively. At a certain time  $t^*$ , these quantities have the values

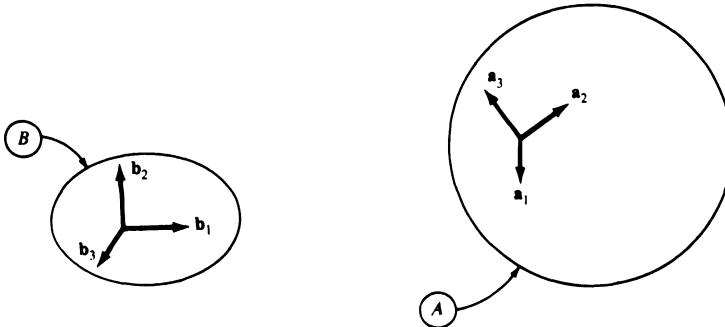


Figure 2.1.1

recorded in Tables 2.1.1 and 2.1.2. The angular velocity of  $B$  in  $A$  at time  $t^*$  is to be determined.

Table 2.1.1

$i$	$\alpha_i$	$\beta_i$	$\gamma_i$
1	0.9363	-0.2896	0.1987
2	0.3130	0.9447	-0.0981
3	-0.1593	0.1540	0.9751

Table 2.1.2

$i$	$\dot{\alpha}_i$ (rad/s)	$\dot{\beta}_i$ (rad/s)	$\dot{\gamma}_i$ (rad/s)
1	-0.0127	-0.0261	0.0216
2	0.0303	-0.0103	-0.0032
3	-0.0148	0.0145	-0.0047

It follows from Eqs. (14) that

$$\mathbf{b}_1 = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 \quad (15)$$

$$\mathbf{b}_2 = \beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_2 + \beta_3 \mathbf{a}_3 \quad (16)$$

and

$$\mathbf{b}_3 = \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \gamma_3 \mathbf{a}_3 \quad (17)$$

Consequently, for all values of the time  $t$ ,

$$\frac{{}^A d\mathbf{b}_1}{dt} = \dot{\alpha}_1 \mathbf{a}_1 + \dot{\alpha}_2 \mathbf{a}_2 + \dot{\alpha}_3 \mathbf{a}_3 \quad (18)$$

$$\frac{{}^A d\mathbf{b}_2}{dt} = \dot{\beta}_1 \mathbf{a}_1 + \dot{\beta}_2 \mathbf{a}_2 + \dot{\beta}_3 \mathbf{a}_3 \quad (19)$$

$$\frac{{}^A d\mathbf{b}_3}{dt} = \dot{\gamma}_1 \mathbf{a}_1 + \dot{\gamma}_2 \mathbf{a}_2 + \dot{\gamma}_3 \mathbf{a}_3 \quad (20)$$

and

$$\begin{aligned} {}^A \boldsymbol{\omega}^B = & \underset{(1)}{\mathbf{b}_1} (\underset{(19,17)}{\dot{\beta}_1 \gamma_1} + \underset{(20,15)}{\dot{\beta}_2 \gamma_2} + \dot{\beta}_3 \gamma_3) + \underset{(20,15)}{\mathbf{b}_2} (\dot{\gamma}_1 \alpha_1 + \dot{\gamma}_2 \alpha_2 + \dot{\gamma}_3 \alpha_3) \\ & + \underset{(18,16)}{\mathbf{b}_3} (\dot{\alpha}_1 \beta_1 + \dot{\alpha}_2 \beta_2 + \dot{\alpha}_3 \beta_3) \end{aligned} \quad (21)$$

Thus, at time  $t^*$ , Eq. (21) together with Tables 2.1.1 and 2.1.2 yields

$${}^A \boldsymbol{\omega}^B = 0.010 \mathbf{b}_1 + 0.020 \mathbf{b}_2 + 0.030 \mathbf{b}_3 \quad \text{rad/s} \quad (22)$$

In Fig. 2.1.2,  $B$  represents a door supported by hinges in a room  $A$ . Mutually perpendicular unit vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  are fixed in  $A$ , with  $\mathbf{a}_3$  parallel to the axis of the hinges, and mutually perpendicular unit vectors  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are fixed in  $B$ , with  $\mathbf{b}_3 = \mathbf{a}_3$ . If  $\theta$  is the radian measure of the angle between  $\mathbf{a}_1$  and  $\mathbf{b}_1$ , as shown in Fig. 2.1.2, then  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  and  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are related to each other as indicated in Table 2.1.3.

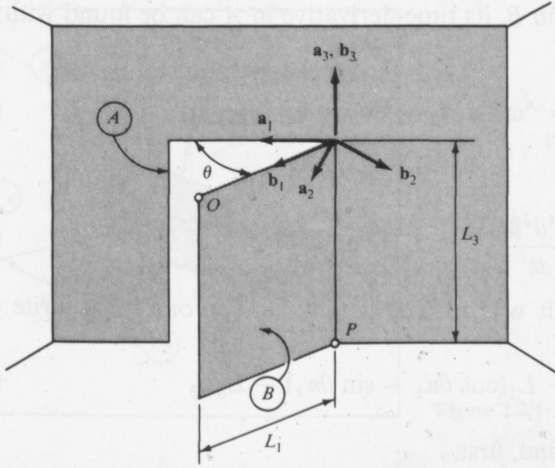


Figure 2.1.2

Table 2.1.3

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$
$\mathbf{a}_1$	$\cos \theta$	$-\sin \theta$	0
$\mathbf{a}_2$	$\sin \theta$	$\cos \theta$	0
$\mathbf{a}_3$	0	0	1

The angular velocity of  $B$  in  $A$ ,  ${}^A\boldsymbol{\omega}^B$ , found with the aid of Eq. (1) and Table 2.1.3, is given by†

$${}^A\boldsymbol{\omega}^B = \dot{\theta}\mathbf{b}_3 \quad (23)$$

and the utility of Eq. (2) becomes apparent when one seeks to find, for example, the second time-derivative in  $A$  of the position vector from the point  $O$  shown in Fig. 2.1.2 to the point  $P$ , that is, of the vector  $\boldsymbol{\beta}$  given by

$$\boldsymbol{\beta} = -L_1\mathbf{b}_1 - L_3\mathbf{b}_3 \quad (24)$$

For, using Eq. (2), one immediately has

$$\frac{{}^Ad\boldsymbol{\beta}}{dt} \underset{(2)}{\underset{(23)}} = \dot{\theta}\mathbf{b}_3 \times (-L_1\mathbf{b}_1 - L_3\mathbf{b}_3) \underset{(24)}{=} -L_1\dot{\theta}\mathbf{b}_2 \quad (25)$$

so that one can write

$$\frac{{}^Ad^2\boldsymbol{\beta}}{dt^2} = \frac{{}^Ad}{dt} \left( \frac{{}^Ad\boldsymbol{\beta}}{dt} \right) \underset{(25)}{=} -L_1\ddot{\theta}\mathbf{b}_2 - L_1\dot{\theta} \frac{{}^Ad\mathbf{b}_2}{dt} \quad (26)$$

† Use of the theorem stated in Sec. 2.2 allows one to write Eq. (23) by inspection.

Since  $\mathbf{b}_2$  is a vector fixed in  $B$ , its time-derivative in  $A$  can be found with the aid of Eq. (2); that is,

$$\frac{{}^A d\mathbf{b}_2}{dt} \underset{(2)}{=} {}^A\boldsymbol{\omega}^B \times \mathbf{b}_2 = \underset{(23)}{\dot{\theta}\mathbf{b}_3} \times \mathbf{b}_2 = -\dot{\theta}\mathbf{b}_1 \quad (27)$$

Consequently,

$$\frac{{}^A d^2\boldsymbol{\beta}}{dt^2} \underset{(26,27)}{=} L_1(\dot{\theta}^2\mathbf{b}_1 - \ddot{\theta}\mathbf{b}_2) \quad (28)$$

To obtain the same result without the use of Eq. (2), one must write (see Table 2.1.3)

$$\boldsymbol{\beta} \underset{(24)}{=} -L_1(\cos\theta\mathbf{a}_1 + \sin\theta\mathbf{a}_2) - L_3\mathbf{a}_3 \quad (29)$$

and then differentiate to find, first,

$$\begin{aligned} \frac{{}^A d\boldsymbol{\beta}}{dt} \underset{(29)}{=} & -L_1(-\dot{\theta}\sin\theta\mathbf{a}_1 + \dot{\theta}\cos\theta\mathbf{a}_2) \\ & = L_1\dot{\theta}(\sin\theta\mathbf{a}_1 - \cos\theta\mathbf{a}_2) \end{aligned} \quad (30)$$

and, next,

$$\frac{{}^A d^2\boldsymbol{\beta}}{dt^2} \underset{(30)}{=} L_1[\ddot{\theta}(\sin\theta\mathbf{a}_1 - \cos\theta\mathbf{a}_2) + \dot{\theta}(\dot{\theta}\cos\theta\mathbf{a}_1 + \dot{\theta}\sin\theta\mathbf{a}_2)] \quad (31)$$

after which one arrives at Eq. (28) by noting that (see Table 2.1.3)

$$\sin\theta\mathbf{a}_1 - \cos\theta\mathbf{a}_2 = -\mathbf{b}_2 \quad (32)$$

while

$$\cos\theta\mathbf{a}_1 + \sin\theta\mathbf{a}_2 = \mathbf{b}_1 \quad (33)$$

In more complex situations, that is, when the motion of  $B$  in  $A$  is more complicated than that of a door  $B$  in a room  $A$ , the use of the angular velocity vector as an “operator” which, through cross-multiplication, produces time-derivatives, is all the more advantageous.

## 2.2 SIMPLE ANGULAR VELOCITY

When a rigid body  $B$  moves in a reference frame  $A$  in such a way that there exists throughout some time interval a unit vector  $\mathbf{k}$  whose orientation in both  $A$  and  $B$  is independent of the time  $t$ , then  $B$  is said to have a *simple angular velocity* in  $A$  throughout this time interval, and this angular velocity can be expressed as

$${}^A\boldsymbol{\omega}^B = \omega\mathbf{k} \quad (1)$$

with  $\omega$  defined as

$$\omega \triangleq \dot{\theta} \quad (2)$$

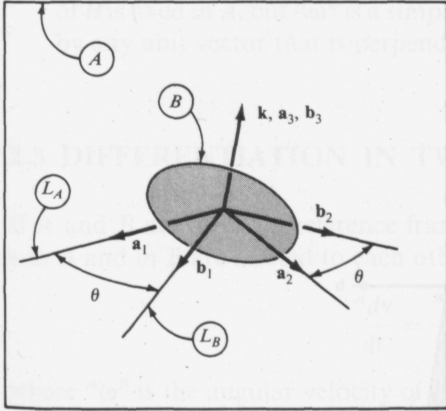


Figure 2.2.1

where  $\theta$  is the radian measure of the angle between a line  $L_A$  whose orientation is fixed in  $A$  and a line  $L_B$  similarly fixed in  $B$  (see Fig. 2.2.1), both lines are perpendicular to  $\mathbf{k}$ , and  $\theta$  is regarded as positive when the angle can be generated by a rotation of  $B$  relative to  $A$  during which a right-handed screw rigidly attached to  $B$  and parallel to  $\mathbf{k}$  advances in the direction of  $\mathbf{k}$ . The scalar quantity  $\omega$  is called an *angular speed* of  $B$  in  $A$ . [The indefinite article “an” is used here because, if  $\tilde{\mathbf{k}}$  and  $\tilde{\omega}$  are defined as  $\tilde{\mathbf{k}} \triangleq -\mathbf{k}$  and  $\tilde{\omega} \triangleq -\omega$ , then Eq. (1) can be written  ${}^A\boldsymbol{\omega}^B = \tilde{\omega}\tilde{\mathbf{k}}$  so that  $\tilde{\omega}$  is no less “the” angular speed than is  $\omega$ .]

**Derivation** Let  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  be a right-handed set of mutually perpendicular unit vectors fixed in  $A$ , with  $\mathbf{a}_1$  parallel to line  $L_A$  and  $\mathbf{a}_3 = \mathbf{k}$ , and let  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  be a similar set of unit vectors fixed in  $B$ , with  $\mathbf{b}_1$  parallel to  $L_B$  and  $\mathbf{b}_3 = \mathbf{k}$ . Then

$$\mathbf{b}_1 = \cos \theta \mathbf{a}_1 + \sin \theta \mathbf{a}_2 \quad (3)$$

$$\mathbf{b}_2 = -\sin \theta \mathbf{a}_1 + \cos \theta \mathbf{a}_2 \quad (4)$$

$$\mathbf{b}_3 = \mathbf{a}_3 \quad (5)$$

$$\frac{{}^A d\mathbf{b}_1}{dt} = \dot{\theta}(-\sin \theta \mathbf{a}_1 + \cos \theta \mathbf{a}_2) = \dot{\theta} \mathbf{b}_2 \quad (6)$$

$$\frac{{}^A d\mathbf{b}_2}{dt} = \dot{\theta}(-\cos \theta \mathbf{a}_1 - \sin \theta \mathbf{a}_2) = -\dot{\theta} \mathbf{b}_1 \quad (7)$$

$$\frac{{}^A d\mathbf{b}_3}{dt} = 0 \quad (8)$$

and substitution into Eq. (2.1.1) leads directly to

$${}^A\boldsymbol{\omega}^B = \dot{\theta} \mathbf{b}_3 = \dot{\theta} \mathbf{k} \quad (9)$$

**Example** Simple angular velocities are encountered most frequently in connection with bodies that are meant to rotate relative to each other about axes

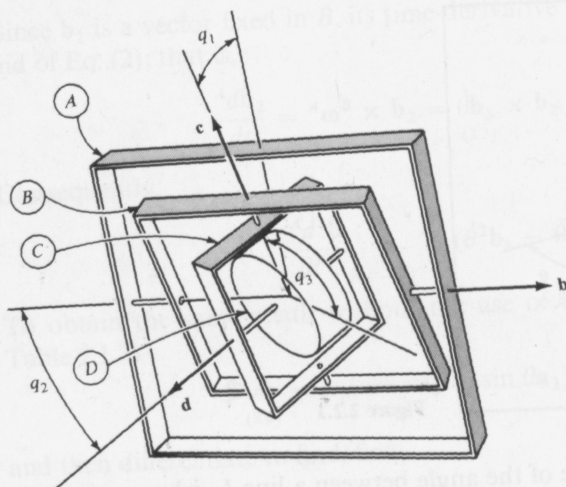


Figure 2.2.2

fixed in the bodies, such as the rotor and the gimbal rings of a gyroscope. This is illustrated in Fig. 2.2.2, where  $A$  designates an aircraft that carries a gyroscope consisting of an outer gimbal  $B$ , an inner gimbal  $C$ , and a rotor  $D$ . Here, the angular velocities of  $B$  in  $A$ ,  $C$  in  $B$ , and  $D$  in  $C$  all are simple angular velocities, and they can be expressed as

$${}^A\omega^B = \dot{q}_1 \mathbf{b} \quad {}^B\omega^C = \dot{q}_2 \mathbf{c} \quad {}^C\omega^D = -\dot{q}_3 \mathbf{d} \quad (10)$$

where  $q_1, q_2, q_3$  measure angles as indicated in Fig. 2.2.2 and  $\mathbf{b}, \mathbf{c}, \mathbf{d}$  are unit vectors directed as shown. (Note that, for example,  ${}^A\omega^C$  and  ${}^B\omega^D$  are *not* simple angular velocities.)

A rigid body  $B$  need not be mounted in bearings fixed in a reference frame  $A$  in order to have a simple angular velocity in  $A$ . Indeed, it is possible for  $B$  to have a simple angular velocity in  $A$  when  $B$  moves in such a way that no point of  $B$  remains fixed in  $A$ . For example, suppose that  $A$  is an aircraft in level flight above a hilly roadway that lies in a vertical plane, as depicted in Fig. 2.2.3, and that  $B$  is an automobile wheel traversing the roadway. No point

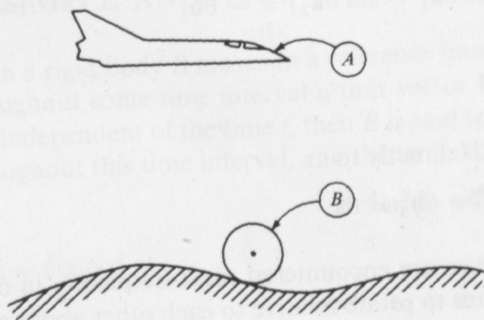


Figure 2.2.3

of  $B$  is fixed in  $A$ , but  ${}^A\omega^B$  is a simple angular velocity, the role of  $\mathbf{k}$  being played by any unit vector that is perpendicular to the middle plane of the wheel.

## 2.3 DIFFERENTIATION IN TWO REFERENCE FRAMES

If  $A$  and  $B$  are any two reference frames, the first time-derivatives of any vector  $\mathbf{v}$  in  $A$  and in  $B$  are related to each other as follows:

$$\frac{{}^Ad\mathbf{v}}{dt} = \frac{{}^Bd\mathbf{v}}{dt} + {}^A\omega^B \times \mathbf{v} \quad (1)$$

where  ${}^A\omega^B$  is the angular velocity of  $B$  in  $A$  (see Sec. 2.1).

**Derivation** With  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  as in Sec. 2.1, let

$$v_i \triangleq \mathbf{v} \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \quad (2)$$

so that [see Eqs. (1.3.1) and (1.3.2)]

$$\mathbf{v} = \sum_{i=1}^3 v_i \mathbf{b}_i \quad (3)$$

Then

$$\begin{aligned} \frac{{}^Ad\mathbf{v}}{dt} &= \sum_{(3)} \frac{dv_i}{dt} \mathbf{b}_i + \sum_{i=1}^3 v_i \frac{{}^Ad\mathbf{b}_i}{dt} \\ &= \frac{{}^Bd\mathbf{v}}{dt} + \sum_{i=1}^3 v_i {}^A\omega^B \times \mathbf{b}_i \\ &= \frac{{}^Bd\mathbf{v}}{dt} + {}^A\omega^B \times \sum_{i=1}^3 v_i \mathbf{b}_i \\ &= \frac{{}^Bd\mathbf{v}}{dt} + {}^A\omega^B \times \mathbf{v} \end{aligned} \quad (4)$$

Equation (1) enables one to find the time-derivative of  $\mathbf{v}$  in  $A$  without having to resolve  $\mathbf{v}$  into components parallel to unit vectors fixed in  $A$ .

**Example** A vector  $\mathbf{H}$ , called the *central angular momentum* of a rigid body  $B$  in a reference frame  $A$ , can be expressed as

$$\mathbf{H} = I_1 \omega_1 \mathbf{b}_1 + I_2 \omega_2 \mathbf{b}_2 + I_3 \omega_3 \mathbf{b}_3 \quad (5)$$

where  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a certain set of mutually perpendicular unit vectors fixed in  $B$ ,  $\omega_j$  is defined as

$$\omega_j \triangleq {}^A\omega^B \cdot \mathbf{b}_j \quad (j = 1, 2, 3) \quad (6)$$

and  $I_1, I_2, I_3$  are constants, called central principal moments of inertia of  $B$ . With the aid of Eq. (1), one can find  $M_1, M_2, M_3$  such that the first time-derivative of  $\mathbf{H}$  in  $A$  is given by

$$\frac{{}^A d\mathbf{H}}{dt} = M_1 \mathbf{b}_1 + M_2 \mathbf{b}_2 + M_3 \mathbf{b}_3 \quad (7)$$

Specifically,

$$\frac{{}^A d\mathbf{H}}{dt} \underset{(1)}{=} \frac{{}^B d\mathbf{H}}{dt} + {}^A \boldsymbol{\omega}^B \times \mathbf{H} \quad (8)$$

$$\frac{{}^B d\mathbf{H}}{dt} \underset{(5)}{=} I_1 \dot{\omega}_1 \mathbf{b}_1 + I_2 \dot{\omega}_2 \mathbf{b}_2 + I_3 \dot{\omega}_3 \mathbf{b}_3 \quad (9)$$

$${}^A \boldsymbol{\omega}^B \underset{(6)}{=} \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 \quad (10)$$

$${}^A \boldsymbol{\omega}^B \times \mathbf{H} \underset{(10,5)}{=} (\omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2) \mathbf{b}_1 + \cdots \quad (11)$$

$$\frac{{}^A d\mathbf{H}}{dt} \underset{(8)}{=} \underset{(9)}{[I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3]} \underset{(11)}{\mathbf{b}_1} + \cdots \quad (12)$$

It follows from Eqs. (7) and (12) that

$$M_1 = I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 \quad (13)$$

$$M_2 = I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 \quad (14)$$

$$M_3 = I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 \quad (15)$$

## 2.4 AUXILIARY REFERENCE FRAMES

The angular velocity of a rigid body  $B$  in a reference frame  $A$  (see Sec. 2.1) can be expressed in the following form involving  $n$  auxiliary reference frames  $A_1, \dots, A_n$ :

$${}^A \boldsymbol{\omega}^B = {}^A \boldsymbol{\omega}^{A_1} + {}^{A_1} \boldsymbol{\omega}^{A_2} + \cdots + {}^{A_{n-1}} \boldsymbol{\omega}^{A_n} + {}^{A_n} \boldsymbol{\omega}^B \quad (1)$$

This relationship, the *addition theorem for angular velocities*, is particularly useful when each term in the right-hand member represents a simple angular velocity (see Sec. 2.2) and can, therefore, be expressed as in Eq. (2.2.1). However, Eq. (1) applies even when one or more of  ${}^A \boldsymbol{\omega}^{A_1}, \dots, {}^{A_n} \boldsymbol{\omega}^B$  are not simple angular velocities.

The reference frames  $A_1, \dots, A_n$  may or may not correspond to actual rigid bodies. Frequently, such reference frames are introduced as aids in analysis, but have no physical counterparts.

When the angular velocity of  $B$  in  $A$  is resolved into components (that is, when  ${}^A \boldsymbol{\omega}^B$  is expressed as the sum of a number of vectors), these components may always be regarded as angular velocities of certain bodies in certain reference frames. Indeed, Eq. (1) represents precisely such a resolution of  ${}^A \boldsymbol{\omega}^B$  into components. In no case, however, are these components themselves angular velocities of



$B$  in  $A$ , for there exists at any one instant only one angular velocity of  $B$  in  $A$ . In other words,  $B$  cannot possess simultaneously† several angular velocities in  $A$ .

**Derivation** For any vector  $\beta$  fixed in  $B$ ,

$$\frac{{}^A d\beta}{dt} \underset{(2.1.2)}{=} {}^A \omega^B \times \beta \quad (2)$$

$$\frac{{}^{A_1} d\beta}{dt} \underset{(2.1.2)}{=} {}^{A_1} \omega^B \times \beta \quad (3)$$

and

$$\frac{{}^A d\beta}{dt} \underset{(2.3.1)}{=} \frac{{}^{A_1} d\beta}{dt} + {}^A \omega^{A_1} \times \beta \quad (4)$$

Hence,

$$\underset{(2)}{{}^A \omega^B} \times \beta = \underset{(4)}{{}^{A_1} \omega^B} \times \beta + \underset{(3)}{{}^A \omega^{A_1}} \times \beta \quad (5)$$

Since this equation is satisfied by *every*  $\beta$  fixed in  $B$ , it implies that

$${}^A \omega^B = {}^A \omega^{A_1} + {}^{A_1} \omega^B \quad (6)$$

which shows that Eq. (1) is valid for  $n = 1$ . Proceeding similarly, one can verify that

$${}^{A_1} \omega^B = {}^{A_1} \omega^{A_2} + {}^{A_2} \omega^B \quad (7)$$

and substitution into Eq. (6) then yields

$${}^A \omega^B = {}^A \omega^{A_1} + {}^{A_1} \omega^{A_2} + {}^{A_2} \omega^B \quad (8)$$

which is Eq. (1) with  $n = 2$ . The validity of Eq. (1) for any value of  $n$  thus can be established by applying this procedure a sufficient number of times.

**Example** In Fig. 2.4.1,  $q_1$ ,  $q_2$ , and  $q_3$  denote the radian measures of angles characterizing the orientation of a rigid cone  $B$  in a reference frame  $A$ . These angles are formed by lines described as follows:  $L_1$  and  $L_2$  are perpendicular to each other and fixed in  $A$ ;  $L_3$  is the axis of symmetry of  $B$ ;  $L_4$  is perpendicular to  $L_2$  and intersects  $L_2$  and  $L_3$ ;  $L_5$  is perpendicular to  $L_3$  and intersects  $L_2$  and  $L_3$ ;  $L_6$  is perpendicular to  $L_3$  and is fixed in  $B$ ;  $L_7$  is perpendicular to  $L_2$  and  $L_4$ . To find an expression for the angular velocity of  $B$  in  $A$ , one can designate as  $A_1$  a reference frame in which  $L_2$ ,  $L_4$ , and  $L_7$  are fixed, and as  $A_2$  a reference frame in which  $L_3$ ,  $L_5$ , and  $L_7$  are fixed, observing that  $L_2$

† In the literature, one encounters not infrequently the equation  $\omega = \omega_1 + \omega_2$ , accompanied by a discussion of “simultaneous angular velocities of a rigid body” and/or “the vector character of angular velocity.” Moreover,  $\omega_1$  and  $\omega_2$  often are called angular velocities of  $B$  about certain axes. This leads one to wonder how many such axes exist in a given case, how one can locate them, and so forth. Since the notion of “angular velocity about an axis” serves no useful purpose, it is best simply to dispense with it.

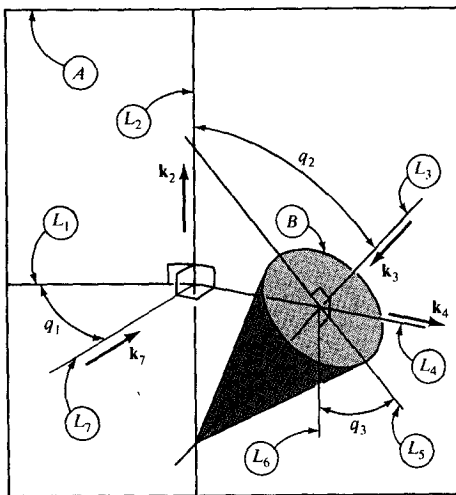


Figure 2.4.1

then is fixed both in  $A$  and  $A_1$ ,  $L_7$  is fixed both in  $A_1$  and  $A_2$ , and  $L_3$  is fixed both in  $A_2$  and  $B$ , so that, in accordance with Eqs. (2.2.1) and (2.2.2), one can write

$${}^A\omega^{A_1} = \dot{q}_1 \mathbf{k}_2 \quad {}^{A_1}\omega^{A_2} = \dot{q}_2 \mathbf{k}_7 \quad {}^{A_2}\omega^B = \dot{q}_3 \mathbf{k}_3 \quad (9)$$

where  $\mathbf{k}_2$ ,  $\mathbf{k}_7$ , and  $\mathbf{k}_3$  are unit vectors directed as shown in Fig. 2.4.1. Substituting from Eq. (9) into Eq. (1) with  $n = 2$ , one arrives at

$${}^A\omega^B = \dot{q}_1 \mathbf{k}_2 + \dot{q}_2 \mathbf{k}_7 + \dot{q}_3 \mathbf{k}_3 \quad (10)$$

## 2.5 ANGULAR ACCELERATION

The *angular acceleration*  ${}^A\alpha^B$  of a rigid body  $B$  in a reference frame  $A$  is defined as the first time-derivative in  $A$  of the angular velocity of  $B$  in  $A$  (see Sec. 2.1):

$${}^A\alpha^B \triangleq \frac{{}^A d {}^A\omega^B}{dt} \quad (1)$$

Since the first time-derivatives of  ${}^A\omega^B$  in  $A$  and in  $B$  are equal to each other, as becomes evident when one replaces  $\mathbf{v}$  in Eq. (2.3.1) with  ${}^A\omega^B$ , Eq. (1) implies that

$${}^A\alpha^B = \frac{{}^B d {}^A\omega^B}{dt} \quad (2)$$

which furnishes a convenient way to find  ${}^A\alpha^B$  when  ${}^A\omega^B$  has been expressed in terms of components parallel to unit vectors fixed in  $B$ .

If  $A_1, \dots, A_n$  are  $n$  auxiliary reference frames,  ${}^A\alpha^B$  is *not*, in general, equal to the sum  ${}^A\alpha^{A_1} + {}^{A_1}\alpha^{A_2} + \dots + {}^{A_n}\alpha^B$ . Thus, Eq. (2.4.1) does not, in general, have an angular acceleration counterpart.

The angular velocity of  $B$  in  $A$  can always be expressed as  ${}^A\omega^B = \omega \mathbf{k}_\omega$ , where  $\mathbf{k}_\omega$  is a unit vector parallel to  ${}^A\omega^B$ ; similarly  ${}^A\alpha^B$  can always be expressed as  ${}^A\alpha^B = \alpha \mathbf{k}_\alpha$ , where  $\mathbf{k}_\alpha$  is a unit vector parallel to  ${}^A\alpha^B$ . In general,  $\mathbf{k}_\omega$  differs from  $\mathbf{k}_\alpha$ , and  $\alpha \neq d\omega/dt$ . But when  $B$  has a simple angular velocity in  $A$  (see Sec. 2.2), and  ${}^A\omega^B$  is expressed as in Eq. (2.2.1), then

$${}^A\alpha^B = \alpha \mathbf{k} \quad (3)$$

where  $\alpha$ , called a *scalar angular acceleration*, is given by

$$\alpha = \frac{d\omega}{dt} \quad (4)$$

**Example** Referring to the example in Sec. 2.4 and to Fig. 2.4.1, one can find an expression for the angular acceleration of the cone  $B$  in reference frame  $A$  as follows:

$$\begin{aligned} {}^A\alpha^B &= \frac{{}^Ad}{dt} (\dot{q}_1 \mathbf{k}_2 + \dot{q}_2 \mathbf{k}_7 + \dot{q}_3 \mathbf{k}_3) \\ &= \ddot{q}_1 \mathbf{k}_2 + \dot{q}_1 \frac{{}^Ad\mathbf{k}_2}{dt} + \ddot{q}_2 \mathbf{k}_7 + \dot{q}_2 \frac{{}^Ad\mathbf{k}_7}{dt} + \ddot{q}_3 \mathbf{k}_3 + \dot{q}_3 \frac{{}^Ad\mathbf{k}_3}{dt} \end{aligned} \quad (5)$$

Since  $\mathbf{k}_2$  is fixed in  $A$ ,

$$\frac{{}^Ad\mathbf{k}_2}{dt} = 0 \quad (6)$$

The unit vector  $\mathbf{k}_7$  is fixed in a reference frame previously called  $A_1$  and having an angular velocity in  $A$  given by

$${}^A\omega^{A_1} = \dot{q}_1 \mathbf{k}_2 \quad (7)$$

Hence,

$$\frac{{}^Ad\mathbf{k}_7}{dt} = {}^A\omega^{A_1} \times \mathbf{k}_7 = \dot{q}_1 \mathbf{k}_2 \times \mathbf{k}_7 \quad (8)$$

Similarly, since  $\mathbf{k}_3$  is fixed in  $B$ ,

$$\frac{{}^Ad\mathbf{k}_3}{dt} = {}^A\omega^B \times \mathbf{k}_3 = \dot{q}_1 \mathbf{k}_2 \times \mathbf{k}_3 + \dot{q}_2 \mathbf{k}_7 \times \mathbf{k}_3 \quad (9)$$

Consequently,

$$\begin{aligned} {}^A\alpha^B &= \ddot{q}_1 \mathbf{k}_2 + 0 + \ddot{q}_2 \mathbf{k}_7 + \dot{q}_2 \dot{q}_1 \mathbf{k}_2 \times \mathbf{k}_7 \\ &\quad + \ddot{q}_3 \mathbf{k}_3 + \dot{q}_3 (\dot{q}_1 \mathbf{k}_2 \times \mathbf{k}_3 + \dot{q}_2 \mathbf{k}_7 \times \mathbf{k}_3) \end{aligned} \quad (10)$$

The angular accelerations of  $A_1$  in  $A$ ,  $A_2$  in  $A_1$ , and  $B$  in  $A_2$  are

$${}^A\alpha^{A_1} = \frac{{}^Ad^A\omega^{A_1}}{dt} = \ddot{q}_1 \mathbf{k}_2 \quad (11)$$

$${}^{A_1}\alpha^{A_2} = \frac{{}^{A_1}d^{A_1}\omega^{A_2}}{dt} = \ddot{q}_2 \mathbf{k}_7 \quad (12)$$

$${}^{A_2}\alpha^B = \frac{{}^{A_2}d^{A_2}\omega^B}{dt} = \ddot{q}_3 \mathbf{k}_3 \quad (13)$$

Hence,

$${}^A\alpha^{A_1} + {}^{A_1}\alpha^{A_2} + {}^{A_2}\alpha^B = \ddot{q}_1 \mathbf{k}_2 + \ddot{q}_2 \mathbf{k}_7 + \ddot{q}_3 \mathbf{k}_3 \neq {}^A\alpha^B \quad (14)$$

## 2.6 VELOCITY AND ACCELERATION

The solution of nearly every problem in dynamics requires the formulation of expressions for velocities and accelerations of points of a system under consideration. At times, the most convenient way to generate the needed expressions is to use the definitions given below. Frequently, however, much labor can be saved by appealing to the theorems† stated in Secs. 2.7 and 2.8.

Let  $\mathbf{p}$  denote the position vector from *any* point  $O$  fixed in a reference frame  $A$  to a point  $P$  moving in  $A$ . The velocity of  $P$  in  $A$  and the acceleration of  $P$  in  $A$ , denoted by  ${}^A\mathbf{v}^P$  and  ${}^A\mathbf{a}^P$ , respectively, are defined as

$${}^A\mathbf{v}^P \triangleq \frac{{}^Ad\mathbf{p}}{dt} \quad (1)$$

and

$${}^A\mathbf{a}^P \triangleq \frac{{}^Ad^A\mathbf{v}^P}{dt} \quad (2)$$

**Example** In Fig. 2.6.1,  $P_1$  and  $P_2$  designate two points connected by a line of length  $L$  and free to move in a plane  $B$  that is rotating at a constant rate  $\omega$  about a line  $Y$  fixed both in  $B$  and in a reference frame  $A$ . The velocities  ${}^A\mathbf{v}^{P_1}$  and  ${}^A\mathbf{v}^{P_2}$  of  $P_1$  and  $P_2$  in  $A$  are to be expressed in terms of the quantities  $q_1, q_2, q_3$ , their time-derivatives  $\dot{q}_1, \dot{q}_2, \dot{q}_3$ , and the mutually perpendicular unit vectors  $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$  shown in Fig. 2.6.1.

† The discussion of velocities and accelerations in Secs. 2.7 and 2.8 involves the concept of angular velocity. Hence, to come into position to present this material without a break in continuity, one must deal with angular velocity before taking up velocity and acceleration. Conversely, as Secs. 2.1–2.4 show, angular velocity can be discussed without any reference to velocity or acceleration. Therefore, it is both natural and advantageous to treat these topics in the order used here, that is, angular velocity before velocity and acceleration, rather than in the reverse order.

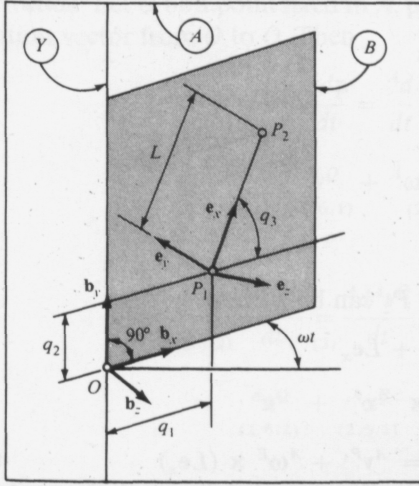


Figure 2.6.1

The point  $O$  shown in Fig. 2.6.1 is a point fixed in  $A$ , and the position vector  $\mathbf{p}_1$  from  $O$  to  $P_1$  can be written

$$\mathbf{p}_1 = q_1 \mathbf{b}_x + q_2 \mathbf{b}_y \quad (3)$$

where  $\mathbf{b}_x$  and  $\mathbf{b}_y$  are unit vectors directed as shown in Fig. 2.6.1. It follows that

$${}^A \mathbf{v}^{P_1} = \frac{{}^A d\mathbf{p}_1}{dt} \underset{(1)}{=} \underset{(2.3.1)}{\frac{{}^B d\mathbf{p}_1}{dt}} + {}^A \boldsymbol{\omega}^B \times \mathbf{p}_1 \quad (4)$$

where

$$\frac{{}^B d\mathbf{p}_1}{dt} \underset{(3)}{=} \dot{q}_1 \mathbf{b}_x + \dot{q}_2 \mathbf{b}_y \quad (5)$$

and

$${}^A \boldsymbol{\omega}^B \times \mathbf{p}_1 = \underset{(2.2.1)}{\omega \mathbf{b}_y} \times \underset{(3)}{(q_1 \mathbf{b}_x + q_2 \mathbf{b}_y)} = -\omega q_1 \mathbf{b}_z \quad (6)$$

so that

$${}^A \mathbf{v}^{P_1} = \underset{(4)}{\dot{q}_1 \mathbf{b}_x} + \underset{(5)}{\dot{q}_2 \mathbf{b}_y} - \underset{(6)}{\omega q_1 \mathbf{b}_z} \quad (7)$$

Since the unit vectors  $\mathbf{b}_x$ ,  $\mathbf{b}_y$ ,  $\mathbf{b}_z$  are related to the unit vectors  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ ,  $\mathbf{e}_z$  as in Table 2.6.1, where  $s_3$  and  $c_3$  stand for  $\sin q_3$  and  $\cos q_3$ , respectively, Eq. (7) is equivalent to

$${}^A \mathbf{v}^{P_1} = (\dot{q}_1 c_3 + \dot{q}_2 s_3) \mathbf{e}_x + (-\dot{q}_1 s_3 + \dot{q}_2 c_3) \mathbf{e}_y - \omega q_1 \mathbf{e}_z \quad (8)$$

which is the desired expression for the velocity of  $P_1$  in  $A$ .

**Table 2.6.1**

	$\mathbf{e}_x$	$\mathbf{e}_y$	$\mathbf{e}_z$
$\mathbf{b}_x$	$c_3$	$-s_3$	0
$\mathbf{b}_y$	$s_3$	$c_3$	0
$\mathbf{b}_z$	0	0	1

The position vector  $\mathbf{p}_2$  from  $O$  to  $P_2$  can be written

$$\mathbf{p}_2 = \mathbf{p}_1 + L\mathbf{e}_x \quad (9)$$

Hence,

$${}^A\mathbf{v}^{P_2} = \frac{{}^A d\mathbf{p}_1}{dt} + \frac{{}^A d}{dt}(L\mathbf{e}_x) = \underset{(1)}{{}^A\mathbf{v}^{P_1}} + \underset{(4)}{{}^A\boldsymbol{\omega}^E} \times \underset{(2.1.2)}{(L\mathbf{e}_x)} \quad (10)$$

where  ${}^A\boldsymbol{\omega}^E$  is the angular velocity in  $A$  of a rigid body  $E$  in which  $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$  are fixed; that is,

$$\begin{aligned} {}^A\boldsymbol{\omega}^E &= {}^A\boldsymbol{\omega}^B + {}^B\boldsymbol{\omega}^E = \underset{(2.4.1)}{\omega\mathbf{b}_y} + \underset{(2.2.1)}{\dot{q}_3\mathbf{b}_z} \\ &= \underset{(\text{Table } 2.6.1)}{\omega s_3\mathbf{e}_x + \omega c_3\mathbf{e}_y} + \dot{q}_3\mathbf{e}_z \end{aligned} \quad (11)$$

so that

$${}^A\boldsymbol{\omega}^E \times (L\mathbf{e}_x) = \underset{(11)}{L(\dot{q}_3\mathbf{e}_y - \omega c_3\mathbf{e}_z)} \quad (12)$$

and

$$\underset{(10,8,12)}{{}^A\mathbf{v}^{P_2}} = (\dot{q}_1 c_3 + \dot{q}_2 s_3)\mathbf{e}_x + (-\dot{q}_1 s_3 + \dot{q}_2 c_3 + L\dot{q}_3)\mathbf{e}_y - \omega(q_1 + Lc_3)\mathbf{e}_z \quad (13)$$

## 2.7 TWO POINTS FIXED ON A RIGID BODY

If  $P$  and  $Q$  are two points fixed on a rigid body  $B$  having an angular velocity  ${}^A\boldsymbol{\omega}^B$  in  $A$ , then the velocity  ${}^A\mathbf{v}^P$  of  $P$  in  $A$  and the velocity  ${}^A\mathbf{v}^Q$  of  $Q$  in  $A$  are related to each other as follows:

$${}^A\mathbf{v}^P = {}^A\mathbf{v}^Q + {}^A\boldsymbol{\omega}^B \times \mathbf{r} \quad (1)$$

where  $\mathbf{r}$  is the position vector from  $Q$  to  $P$ . The relationship between the acceleration  ${}^A\mathbf{a}^P$  of  $P$  in  $A$  and the acceleration  ${}^A\mathbf{a}^Q$  of  $Q$  in  $A$  involves the angular acceleration  ${}^A\boldsymbol{\alpha}^B$  of  $B$  in  $A$  and is given by

$${}^A\mathbf{a}^P = {}^A\mathbf{a}^Q + {}^A\boldsymbol{\omega}^B \times ({}^A\boldsymbol{\omega}^B \times \mathbf{r}) + {}^A\boldsymbol{\alpha}^B \times \mathbf{r} \quad (2)$$

**Derivation** Let  $O$  be a point fixed in  $A$ ,  $\mathbf{p}$  the position vector from  $O$  to  $P$ , and  $\mathbf{q}$  the position vector from  $O$  to  $Q$ . Then

$$\begin{aligned} {}^A\mathbf{v}^P &= \frac{{}^A d\mathbf{p}}{dt} = \frac{{}^A d}{dt}(\mathbf{q} + \mathbf{r}) = \frac{{}^A d\mathbf{q}}{dt} + \frac{{}^A d\mathbf{r}}{dt} \\ &= \underset{(2.6.1)}{{}^A\mathbf{v}^Q} + \underset{(2.1.2)}{{}^A\boldsymbol{\omega}^B \times \mathbf{r}} \end{aligned} \quad (3)$$

and

$$\begin{aligned} {}^A\mathbf{a}^P &= \frac{{}^A d {}^A\mathbf{v}^P}{dt} = \underset{(3)}{\frac{{}^A d {}^A\mathbf{v}^Q}{dt}} + \frac{{}^A d {}^A\boldsymbol{\omega}^B}{dt} \times \mathbf{r} + {}^A\boldsymbol{\omega}^B \times \frac{{}^A d\mathbf{r}}{dt} \\ &= \underset{(2.6.2)}{{}^A\mathbf{a}^Q} + \underset{(2.5.1)}{{}^A\boldsymbol{\alpha}^B \times \mathbf{r}} + \underset{(2.1.2)}{{}^A\boldsymbol{\omega}^B \times ({}^A\boldsymbol{\omega}^B \times \mathbf{r})} \end{aligned} \quad (4)$$

**Example** Since an expression for the velocity of  $P_1$  in  $A$  in the example in Sec. 2.6 is available in Eq. (2.6.7), the acceleration of  $P_1$  in  $A$  can be found most directly by differentiating this expression, which yields

$$\begin{aligned} {}^A\mathbf{a}^{P_1} &= \frac{{}^A d {}^A\mathbf{v}^{P_1}}{dt} = \underset{(2.3.1)}{\frac{{}^B d {}^A\mathbf{v}^{P_1}}{dt}} + {}^A\boldsymbol{\omega}^B \times {}^A\mathbf{v}^{P_1} \\ &= \underset{(2.6.7)}{\ddot{q}_1 \mathbf{b}_x + \ddot{q}_2 \mathbf{b}_y - \omega \dot{q}_1 \mathbf{b}_z} + \underset{(2.2.1)}{\omega \mathbf{b}_y \times {}^A\mathbf{v}^{P_1}} \\ &= \underset{(2.6.7)}{(\ddot{q}_1 - \omega^2 q_1) \mathbf{b}_x + \ddot{q}_2 \mathbf{b}_y - 2\omega \dot{q}_1 \mathbf{b}_z} \end{aligned} \quad (5)$$

In the case of  $P_2$ , it is more convenient to use Eq. (2) together with the result just obtained than it is to differentiate the expression for  ${}^A\mathbf{v}^{P_2}$  available in Eq. (2.6.13). Letting  $P_1$  and  $P_2$  play the parts of  $Q$  and  $P$ , respectively, in Eq. (2), and replacing  $B$  with  $E$ , since  $P_1$  and  $P_2$  are fixed on  $E$ , not on  $B$ , one can write

$$\underset{(2)}{{}^A\mathbf{a}^{P_2}} = \underset{(2)}{{}^A\mathbf{a}^{P_1}} + {}^A\boldsymbol{\omega}^E \times [{}^A\boldsymbol{\omega}^E \times (L\mathbf{e}_x)] + {}^A\boldsymbol{\alpha}^E \times (L\mathbf{e}_x) \quad (6)$$

Now,

$$\begin{aligned} {}^A\boldsymbol{\omega}^E \times [{}^A\boldsymbol{\omega}^E \times (L\mathbf{e}_x)] &= \underset{(2.6.11, 2.6.12)}{L[-(\omega^2 c_3^2 + \dot{q}_3^2)\mathbf{e}_x} \\ &\quad + \omega^2 s_3 c_3 \mathbf{e}_y + \omega \dot{q}_3 s_3 \mathbf{e}_z] \end{aligned} \quad (7)$$

and

$$\underset{(2.5.2)}{{}^A\boldsymbol{\alpha}^E} = \frac{{}^E d {}^A\boldsymbol{\omega}^E}{dt} = \underset{(2.6.11)}{\omega \dot{q}_3 c_3 \mathbf{e}_x - \omega \dot{q}_3 s_3 \mathbf{e}_y + \ddot{q}_3 \mathbf{e}_z} \quad (8)$$

so that

$$\underset{(8)}{{}^A\boldsymbol{\alpha}^E \times (L\mathbf{e}_x)} = L(\ddot{q}_3 \mathbf{e}_y + \omega \dot{q}_3 s_3 \mathbf{e}_z) \quad (9)$$

Substitution from Eqs. (5), (7), and (9) into Eq. (6) yields

$$\begin{aligned} {}^A\mathbf{a}^{P_2} = & (\ddot{q}_1 - \omega^2 q_1)\mathbf{b}_x + \ddot{q}_2\mathbf{b}_y - 2\omega(\dot{q}_1 - L\dot{q}_3 s_3)\mathbf{b}_z \\ & + L[-(\omega^2 c_3^2 + \dot{q}_3^2)\mathbf{e}_x + (\omega^2 s_3 c_3 + \ddot{q}_3)\mathbf{e}_y] \end{aligned} \quad (10)$$

If one wishes to express this vector solely in terms of  $\mathbf{b}_x$ ,  $\mathbf{b}_y$ ,  $\mathbf{b}_z$ , one can refer to Table 2.6.1 to obtain

$$\begin{aligned} {}^A\mathbf{a}^{P_2} = & [\ddot{q}_1 - \omega^2 q_1 - L(\ddot{q}_3 s_3 + \dot{q}_3^2 c_3 + \omega^2 c_3)]\mathbf{b}_x \\ & + [\ddot{q}_2 + L(\ddot{q}_3 c_3 - \dot{q}_3^2 s_3)]\mathbf{b}_y - 2\omega(\dot{q}_1 - L\dot{q}_3 s_3)\mathbf{b}_z \end{aligned} \quad (11)$$

## 2.8 ONE POINT MOVING ON A RIGID BODY

If a point  $P$  is moving on a rigid body  $B$  while  $B$  is moving in a reference frame  $A$ , the velocity  ${}^A\mathbf{v}^P$  of  $P$  in  $A$  is related to the velocity  ${}^B\mathbf{v}^P$  of  $P$  in  $B$  as follows:

$${}^A\mathbf{v}^P = {}^A\mathbf{v}^{\bar{B}} + {}^B\mathbf{v}^P \quad (1)$$

where  ${}^A\mathbf{v}^{\bar{B}}$  denotes the velocity in  $A$  of the point  $\bar{B}$  of  $B$  that coincides with  $P$  at the instant under consideration. The acceleration  ${}^A\mathbf{a}^P$  of  $P$  in  $A$  is given by

$${}^A\mathbf{a}^P = {}^A\mathbf{a}^{\bar{B}} + {}^B\mathbf{a}^P + 2{}^A\boldsymbol{\omega}^B \times {}^B\mathbf{v}^P \quad (2)$$

where  ${}^A\mathbf{a}^{\bar{B}}$  is the acceleration of  $\bar{B}$  in  $A$ ,  ${}^B\mathbf{a}^P$  is the acceleration of  $P$  in  $B$ , and  ${}^A\boldsymbol{\omega}^B$  is the angular velocity of  $B$  in  $A$ . The term  $2{}^A\boldsymbol{\omega}^B \times {}^B\mathbf{v}^P$  is referred to as “Coriolis acceleration.”

**Derivation** Let  $\tilde{A}$  be a point fixed in  $A$ ,  $\tilde{B}$  a point fixed in  $B$ ,  $\mathbf{p}$  the position vector from  $\tilde{A}$  to  $P$ ,  $\mathbf{q}$  the position vector from  $\tilde{B}$  to  $P$ , and  $\mathbf{r}$  the position vector from  $\tilde{A}$  to  $\tilde{B}$ , as shown in Fig. 2.8.1. Then, in accordance with Eq. (2.6.1), the velocities  ${}^A\mathbf{v}^P$ ,  ${}^B\mathbf{v}^P$ , and  ${}^A\mathbf{v}^{\tilde{B}}$  are given by

$${}^A\mathbf{v}^P = \frac{{}^A d\mathbf{p}}{dt} \quad (3)$$

$${}^B\mathbf{v}^P = \frac{{}^B d\mathbf{q}}{dt} \quad (4)$$

$${}^A\mathbf{v}^{\tilde{B}} = \frac{{}^A d\mathbf{r}}{dt} \quad (5)$$

As can be seen in Fig. 2.8.1,

$$\mathbf{p} = \mathbf{r} + \mathbf{q} \quad (6)$$



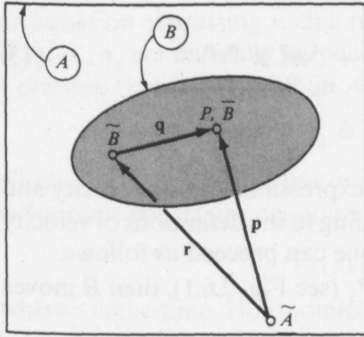


Figure 2.8.1

Hence,

$$\begin{aligned}
 {}^A\mathbf{v}^P &= \frac{{}^A d\mathbf{r}}{dt} + \frac{{}^A d\mathbf{q}}{dt} \\
 &\stackrel{(3,6)}{=} {}^A\mathbf{v}^{\tilde{B}} + \frac{{}^B d\mathbf{q}}{dt} + {}^A\boldsymbol{\omega}^B \times \mathbf{q} \\
 &\stackrel{(5)}{=} {}^A\mathbf{v}^{\tilde{B}} + {}^B\mathbf{v}^P + {}^A\boldsymbol{\omega}^B \times \mathbf{q} \\
 &\stackrel{(4)}{=} {}^A\mathbf{v}^{\tilde{B}} + {}^B\mathbf{v}^P + {}^A\boldsymbol{\omega}^B \times \mathbf{q}
 \end{aligned} \tag{7}$$

$\tilde{B}$  may always be taken as  $\bar{B}$ , that is, as the point of  $B$  that coincides with  $P$  at the instant under consideration. In that event,

$${}^A\mathbf{v}^{\tilde{B}} = {}^A\mathbf{v}^{\bar{B}} \quad \mathbf{q} = 0 \tag{8}$$

and substitution from Eqs. (8) into Eq. (7) leads to Eq. (1).

Referring to Eq. (2.6.2), one can write

$${}^A\mathbf{a}^P = \frac{{}^A d{}^A\mathbf{v}^P}{dt} \tag{9}$$

$${}^B\mathbf{a}^P = \frac{{}^B d{}^B\mathbf{v}^P}{dt} \tag{10}$$

$${}^A\mathbf{a}^{\tilde{B}} = \frac{{}^A d{}^A\mathbf{v}^{\tilde{B}}}{dt} \tag{11}$$

Substitution from Eq. (7) into Eq. (9) yields

$$\begin{aligned}
 {}^A\mathbf{a}^P &= \frac{{}^A d{}^A\mathbf{v}^{\tilde{B}}}{dt} + \frac{{}^A d{}^B\mathbf{v}^P}{dt} + \frac{{}^A d{}^A\boldsymbol{\omega}^B}{dt} \times \mathbf{q} + {}^A\boldsymbol{\omega}^B \times \frac{{}^A d\mathbf{q}}{dt} \\
 &\stackrel{(11)}{=} {}^A\mathbf{a}^{\tilde{B}} + \frac{{}^B d{}^B\mathbf{v}^P}{dt} + {}^A\boldsymbol{\omega}^B \times {}^B\mathbf{v}^P + {}^A\boldsymbol{\alpha}^B \times \mathbf{q} \\
 &\quad + {}^A\boldsymbol{\omega}^B \times \left( \frac{{}^B d\mathbf{q}}{dt} + {}^A\boldsymbol{\omega}^B \times \mathbf{q} \right)
 \end{aligned} \tag{12}$$

(2.3.1)

and, if  $\bar{B}$  is once again taken as  $\bar{B}$ , this reduces to

$${}^A\mathbf{a}^P = \underset{(12)}{{}^A\mathbf{a}^{\bar{B}}} + \underset{(10)}{{}^B\mathbf{a}^P} + \underset{(4)}{{}^A\boldsymbol{\omega}^B \times {}^B\mathbf{v}^P} + \underset{(4)}{{}^A\boldsymbol{\omega}^B \times {}^B\mathbf{v}^P} \quad (13)$$

in agreement with Eq. (2).

**Example** In the examples in Secs. 2.6 and 2.7, expressions for the velocity and the acceleration of  $P_1$  in  $A$  were found by appealing to the definitions of velocity and acceleration, respectively. Alternatively, one can proceed as follows.

If  $\bar{B}$  is the point of  $B$  that coincides with  $P_1$  (see Fig. 2.6.1), then  $\bar{B}$  moves on a circle of radius  $q_1$ , and

$${}^A\mathbf{v}^{\bar{B}} = -\omega q_1 \mathbf{b}_z \quad (14)$$

while (remember that  $\omega$  is a constant)

$${}^A\mathbf{a}^{\bar{B}} = -\omega^2 q_1 \mathbf{b}_x \quad (15)$$

The velocity and acceleration of  $P_1$  in  $B$  are

$${}^B\mathbf{v}^{P_1} = \dot{q}_1 \mathbf{b}_x + \dot{q}_2 \mathbf{b}_y \quad (16)$$

and

$${}^B\mathbf{a}^{P_1} = \ddot{q}_1 \mathbf{b}_x + \ddot{q}_2 \mathbf{b}_y \quad (17)$$

respectively, and the angular velocity of  $B$  in  $A$  is given by

$$\underset{(2.2.1)}{{}^A\boldsymbol{\omega}^B} = \omega \mathbf{b}_y \quad (18)$$

Thus,

$$\underset{(1)}{{}^A\mathbf{v}^{P_1}} = \underset{(14)}{-\omega q_1 \mathbf{b}_z} + \underset{(16)}{\dot{q}_1 \mathbf{b}_x} + \dot{q}_2 \mathbf{b}_y \quad (19)$$

in agreement with Eq. (2.6.7), and

$$\underset{(2)}{{}^A\mathbf{a}^{P_1}} = \underset{(15)}{-\omega^2 q_1 \mathbf{b}_x} + \underset{(17)}{\ddot{q}_1 \mathbf{b}_x} + \ddot{q}_2 \mathbf{b}_y - \underset{(18,16)}{2\omega \dot{q}_1 \mathbf{b}_z} \quad (20)$$

which is the result previously recorded as Eq. (2.7.5).

## 2.9 CONFIGURATION CONSTRAINTS

The *configuration* of a set  $S$  of  $v$  particles  $P_1, \dots, P_v$  in a reference frame  $A$  is known whenever the position vector of each particle relative to a point fixed in  $A$  is known. Thus,  $v$  vector quantities, or, equivalently,  $3v$  scalar quantities, are required for the specification of the configuration of  $S$  in  $A$ .

If the motion of  $S$  is affected by the presence of bodies that come into contact with one or more of  $P_1, \dots, P_v$ , restrictions are imposed on the positions that the affected particles may occupy, and  $S$  is said to be subject to *configuration constraints*;

an equation expressing such a restriction is called a *holonomic constraint equation*. If  $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$  are mutually perpendicular unit vectors fixed in  $A$ , and  $x_i, y_i, z_i$ , called *Cartesian coordinates* of  $P_i$  in  $A$ , are defined as

$$x_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_x \quad y_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_y \quad z_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_z \quad (i = 1, \dots, v) \quad (1)$$

where  $\mathbf{p}_i$  is the position vector from a point  $O$  fixed in  $A$  to the point  $P_i$ , then a holonomic constraint equation has the form

$$f(x_1, y_1, z_1, \dots, x_v, y_v, z_v, t) = 0 \quad (2)$$

where  $t$  is the time. Holonomic constraint equations are classified as *rheonomic* or *scleronomic*, according to whether the function  $f$  does, or does not, contain  $t$  explicitly.

**Example** Figure 2.9.1 shows two small blocks,  $P_1$  and  $P_2$ , connected by a thin rod  $R$  of length  $L$ , and constrained to remain between two parallel panes of glass that are attached to each other, forming a rigid body  $B$ . This body is made to rotate at a constant rate  $\omega$  about a line  $Y$  fixed both in  $B$  and in a reference frame  $A$ . Treating  $P_1$  and  $P_2$  as a set  $S$  of two particles, and letting  $\mathbf{p}_1$  and  $\mathbf{p}_2$  be their position vectors relative to the point  $O$  shown in Fig. 2.9.1, one can express  $\mathbf{p}_1$  and  $\mathbf{p}_2$  as

$$\mathbf{p}_i = x_i \mathbf{a}_x + y_i \mathbf{a}_y + z_i \mathbf{a}_z \quad (i = 1, 2) \quad (3)$$

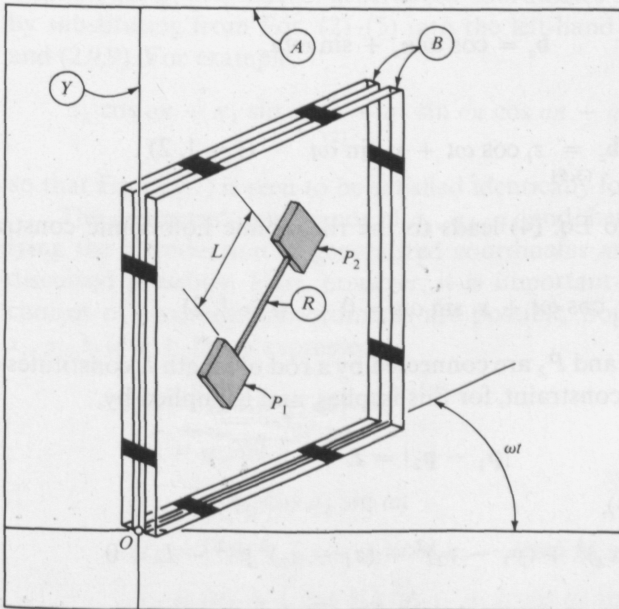


Figure 2.9.1

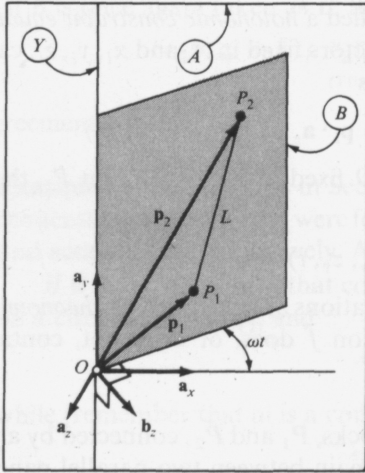


Figure 2.9.2

where  $\mathbf{a}_x$ ,  $\mathbf{a}_y$ ,  $\mathbf{a}_z$  are mutually perpendicular unit vectors fixed in  $A$ , as shown in Fig. 2.9.2. The requirement that  $P_1$  and  $P_2$  remain at all times between the two panes of glass is fulfilled if, and only if,

$$\mathbf{p}_i \cdot \mathbf{b}_z = 0 \quad (i = 1, 2) \quad (4)$$

where  $\mathbf{b}_z$  is a unit vector normal to the plane determined by the panes of glass, as indicated in Fig. 2.9.2. Now,

$$\mathbf{b}_z = \cos \omega t \mathbf{a}_z + \sin \omega t \mathbf{a}_x \quad (5)$$

Hence,

$$\mathbf{p}_i \cdot \mathbf{b}_z \underset{(3,5)}{=} z_i \cos \omega t + x_i \sin \omega t \quad (i = 1, 2) \quad (6)$$

and substitution into Eq. (4) leads to the rheonomic holonomic constraint equations

$$z_i \cos \omega t + x_i \sin \omega t = 0 \quad (i = 1, 2) \quad (7)$$

The fact that  $P_1$  and  $P_2$  are connected by a rod of length  $L$  constitutes one more configuration constraint, for this implies, and is implied by,

$$|\mathbf{p}_1 - \mathbf{p}_2| = L \quad (8)$$

or, in view of Eqs. (3),

$$[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} - L = 0 \quad (9)$$

Since in this equation, in contrast with Eq. (7),  $t$  does not appear explicitly, Eq. (9) is a scleronomic holonomic constraint equation.

## 2.10 GENERALIZED COORDINATES

When a set  $S$  of  $v$  particles  $P_1, \dots, P_v$  is subject to constraints (see Sec. 2.9) represented by  $M$  holonomic constraint equations, only

$$n \triangleq 3v - M \quad (1)$$

of the  $3v$  Cartesian coordinates  $x_i, y_i, z_i$  ( $i = 1, \dots, v$ ) of  $S$  in a reference frame  $A$  are independent of each other. Under these circumstances one can express each of  $x_i, y_i, z_i$  ( $i = 1, \dots, v$ ) as a single-valued function of the time  $t$  and  $n$  functions of  $t$ , say,  $q_1(t), \dots, q_n(t)$ , in such a way that the constraint equations are satisfied identically for all values of  $t$  and  $q_1, \dots, q_n$  in a given domain. The quantities  $q_1, \dots, q_n$  are called *generalized coordinates* for  $S$  in  $A$ .

**Example** For the set  $S$  in the example in Sec. 2.9,  $v = 2$  and  $M = 3$ . Hence  $n = 3$ . Three generalized coordinates for  $S$  in  $A$  may be introduced by expressing  $x_i, y_i, z_i$  ( $i = 1, 2$ ) as

$$x_1 = q_1 \cos \omega t \quad y_1 = q_2 \quad z_1 = -q_1 \sin \omega t \quad (2)$$

$$x_2 = (q_1 + L \cos q_3) \cos \omega t \quad (3)$$

$$y_2 = q_2 + L \sin q_3 \quad (4)$$

$$z_2 = -(q_1 + L \cos q_3) \sin \omega t \quad (5)$$

That  $q_1, q_2, q_3$  are, indeed, generalized coordinates of  $S$  in  $A$  may be verified by substituting from Eqs. (2)–(5) into the left-hand members of Eqs. (2.9.7) and (2.9.9). For example,

$$z_1 \cos \omega t + x_1 \sin \omega t \underset{(2)}{=} -q_1 \sin \omega t \cos \omega t + q_1 \cos \omega t \sin \omega t \equiv 0 \quad (6)$$

so that Eq. (2.9.7) is seen to be satisfied identically for  $i = 1$ .

The geometric significance of  $q_1, q_2, q_3$  and, hence, the rationale underlying the introduction of generalized coordinates as in Eqs. (2)–(5), will be discussed presently. First, however, it is important to point out that other choices of generalized coordinates are possible. Suppose, for example, that  $x_i, y_i, z_i$  ( $i = 1, 2$ ) are expressed as

$$x_1 = q_1 \cos q_2 \cos \omega t \quad (7)$$

$$y_1 = q_1 \sin q_2 \quad (8)$$

$$z_1 = -q_1 \cos q_2 \sin \omega t \quad (9)$$

$$x_2 = [(q_1 + L \cos q_3) \cos q_2 - L \sin q_2 \sin q_3] \cos \omega t \quad (10)$$

$$y_2 = (q_1 + L \cos q_3) \sin q_2 + L \cos q_2 \sin q_3 \quad (11)$$

$$z_2 = -[(q_1 + L \cos q_3) \cos q_2 - L \sin q_2 \sin q_3] \sin \omega t \quad (12)$$

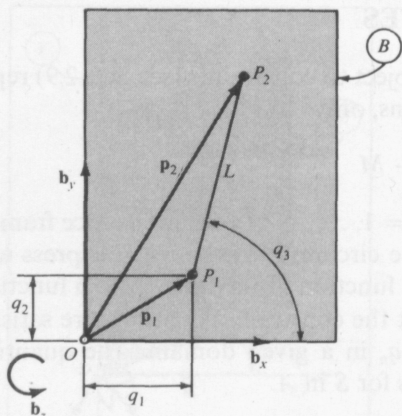


Figure 2.10.1

Then Eqs. (2.9.7) and (2.9.9) are satisfied identically, which means that, once again,  $q_1, q_2, q_3$  are generalized coordinates of  $S$  in  $A$ .

The generalized coordinates introduced in Eqs. (2)–(5) may be regarded as measures of two distances and an angle as indicated in Fig. 2.10.1. This may be seen as follows. If  $\mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z$  are mutually perpendicular unit vectors directed as in Fig. 2.10.1, then  $\mathbf{p}_1$  and  $\mathbf{p}_2$  can be written

$$\mathbf{p}_1 = q_1 \mathbf{b}_x + q_2 \mathbf{b}_y \quad \mathbf{p}_2 = \mathbf{p}_1 + L \cos q_3 \mathbf{b}_x + L \sin q_3 \mathbf{b}_y \quad (13)$$

and

$$x_1 \stackrel{\Delta}{=} \underset{(2.9.1)}{\mathbf{p}_1 \cdot \mathbf{a}_x} = \underset{(13)}{q_1 \mathbf{b}_x \cdot \mathbf{a}_x + q_2 \mathbf{b}_y \cdot \mathbf{a}_x} \quad (14)$$

$$y_1 \stackrel{\Delta}{=} \underset{(2.9.1)}{\mathbf{p}_1 \cdot \mathbf{a}_y} = \underset{(13)}{q_1 \mathbf{b}_x \cdot \mathbf{a}_y + q_2 \mathbf{b}_y \cdot \mathbf{a}_y} \quad (15)$$

$$z_1 \stackrel{\Delta}{=} \underset{(2.9.1)}{\mathbf{p}_1 \cdot \mathbf{a}_z} = \underset{(13)}{q_1 \mathbf{b}_x \cdot \mathbf{a}_z + q_2 \mathbf{b}_y \cdot \mathbf{a}_z} \quad (16)$$

The dot-products appearing in these equations are evaluated most conveniently by referring to Table 2.10.1, which is a concise way of stating the six equations that relate  $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$  to  $\mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z$ . Thus one finds that Eqs. (14)–(16) give way to precisely Eqs. (2).

Table 2.10.1

	$\mathbf{b}_x$	$\mathbf{b}_y$	$\mathbf{b}_z$
$\mathbf{a}_x$	$\cos \omega t$	0	$\sin \omega t$
$\mathbf{a}_y$	0	1	0
$\mathbf{a}_z$	$-\sin \omega t$	0	$\cos \omega t$

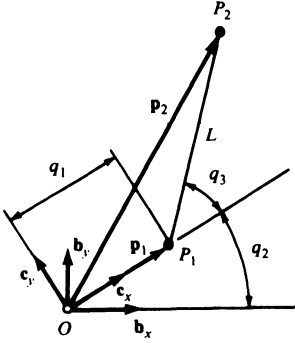


Figure 2.10.2

Similarly,

$$\begin{aligned}
 x_2 &\stackrel{\Delta}{=} \mathbf{p}_2 \cdot \mathbf{a}_x = \mathbf{p}_1 \cdot \mathbf{a}_x + L \cos q_3 \mathbf{b}_x \cdot \mathbf{a}_x + L \sin q_3 \mathbf{b}_y \cdot \mathbf{a}_x \\
 &\quad (2.9.1) \qquad (13) \\
 &= q_1 \cos \omega t + L \cos q_3 \cos \omega t \\
 &\quad (14, \text{Table 2.10.1})
 \end{aligned} \tag{17}$$

which is the same as Eq. (3), and when  $y_2$  and  $z_2$  are formed correspondingly, Eqs. (4) and (5) are recovered.

In the case of the generalized coordinates appearing in Eqs. (7)–(12),  $q_1$  may be interpreted as the distance from  $O$  to  $P_1$ , and  $q_2$  and  $q_3$  as the angles indicated in Fig. 2.10.2. To see this, express  $\mathbf{p}_1$  and  $\mathbf{p}_2$  as

$$\mathbf{p}_1 = q_1 \mathbf{c}_x \quad \mathbf{p}_2 = \mathbf{p}_1 + L \cos q_3 \mathbf{c}_x + L \sin q_3 \mathbf{c}_y \tag{18}$$

where  $\mathbf{c}_x$  and  $\mathbf{c}_y$  are the unit vectors shown in Fig. 2.10.2, and form  $x_i, y_i, z_i$  ( $i = 1, 2$ ) in accordance with Eqs. (2.9.1).

## 2.11 NUMBER OF GENERALIZED COORDINATES

The number  $n$  of generalized coordinates of a set  $S$  of  $v$  particles in a reference frame  $A$  (see Sec. 2.10) is the smallest number of scalar quantities such that to every assignment of values to these quantities and the time  $t$  (within a domain of interest) there corresponds a definite admissible configuration of  $S$  in  $A$ . Frequently, one can find  $n$  by inspection rather than by determining the number  $M$  of holonomic constraint equations (see Sec. 2.9) and then subtracting  $M$  from  $3v$ . For example, suppose that  $S$  consists of  $v$  particles  $P_1, \dots, P_v$  forming a rigid body  $B$  that is free to move in  $A$ . Then there corresponds a definite admissible configuration of  $S$  in  $A$  to every assignment of values to three Cartesian coordinates of one particle of  $B$  and three angles that characterize the orientation of  $B$  in  $A$ . Hence,  $n = 6$ . The same conclusion is obtained formally by letting  $\mathbf{p}_1, \dots, \mathbf{p}_v$  be the position vectors from a point fixed in  $A$  to  $P_1, \dots, P_v$ , respectively, and noting that rigidity can be ensured by letting  $P_1, P_2$ , and  $P_3$  be noncollinear particles and requiring

(1) that the distances between  $P_1$  and  $P_2$ ,  $P_2$  and  $P_3$ , and  $P_3$  and  $P_1$  remain constant, so that

$$(\mathbf{p}_1 - \mathbf{p}_2)^2 = c_1 \quad (1)$$

$$(\mathbf{p}_2 - \mathbf{p}_3)^2 = c_2 \quad (2)$$

$$(\mathbf{p}_3 - \mathbf{p}_1)^2 = c_3 \quad (3)$$

where  $c_1, c_2, c_3$  are constants, and (2) that the distances between each of the remaining  $v - 3$  particles and each of  $P_1, P_2$ , and  $P_3$  remain constant, that is,

$$(\mathbf{p}_i - \mathbf{p}_1)^2 = c_{i1} \quad (i = 4, \dots, v) \quad (4)$$

$$(\mathbf{p}_i - \mathbf{p}_2)^2 = c_{i2} \quad (i = 4, \dots, v) \quad (5)$$

$$(\mathbf{p}_i - \mathbf{p}_3)^2 = c_{i3} \quad (i = 4, \dots, v) \quad (6)$$

where  $c_{ij}$  ( $i = 4, \dots, v; j = 1, 2, 3$ ) are constants. The number  $M$  of holonomic constraint equations is thus given by

$$M = 3 + 3(v - 3) = 3v - 6 \quad (7)$$

and it follows that

$$n = 3v - M = 3v - (3v - 6) = 6 \quad (8)$$

(2.10.1) (7)

## 2.12 GENERALIZED SPEEDS

As will be seen presently, expressions for angular velocities of rigid bodies and velocities of points of a system  $S$  whose configuration in a reference frame  $A$  is characterized by  $n$  generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10) can be brought into particularly advantageous forms through the introduction of  $n$  quantities  $u_1, \dots, u_n$ , called *generalized speeds* for  $S$  in  $A$ , these being quantities defined by equations of the form

$$u_r \triangleq \sum_{s=1}^n Y_{rs} \dot{q}_s + Z_r \quad (r = 1, \dots, n) \quad (1)$$

where  $Y_{rs}$  and  $Z_r$  are functions of  $q_1, \dots, q_n$ , and the time  $t$ . These functions must be chosen such that Eqs. (1) can be solved uniquely for  $\dot{q}_1, \dots, \dot{q}_n$ . Equations (1) are called *kinematical differential equations* for  $S$  in  $A$ .

**Example** Letting  $S$  be the set of two particles considered in the example in Sec. 2.9, and using as generalized coordinates the quantities  $q_1, q_2, q_3$  indicated in Fig. 2.10.1, one may define three generalized speeds as

$$u_1 \triangleq \dot{q}_1 \cos \omega t - \omega q_1 \sin \omega t \quad u_2 \triangleq \dot{q}_2 \quad u_3 \triangleq \dot{q}_3 \quad (2)$$



In that event, the functions  $Y_{rs}$  and  $Z_r$  ( $r, s = 1, 2, 3$ ) of Eq. (1) are

$$Y_{11} = \cos \omega t \quad Y_{12} = Y_{13} = 0 \quad Z_1 = -\omega q_1 \sin \omega t \quad (3)$$

$$Y_{21} = 0 \quad Y_{22} = 1 \quad Y_{23} = 0 \quad Z_2 = 0 \quad (4)$$

$$Y_{31} = 0 \quad Y_{32} = 0 \quad Y_{33} = 1 \quad Z_3 = 0 \quad (5)$$

and, solved for  $\dot{q}_1, \dot{q}_2, \dot{q}_3$ , Eqs. (2) yield

$$\dot{q}_1 = u_1 \sec \omega t + \omega q_1 \tan \omega t \quad \dot{q}_2 = u_2 \quad \dot{q}_3 = u_3 \quad (6)$$

Since  $\sec \omega t$  and  $\tan \omega t$  become infinite whenever  $\omega t$  is equal to an odd multiple of  $\pi/2$  rad, Eqs. (2) furnish acceptable definitions of  $u_1, u_2, u_3$  except when  $\omega t$  takes on one of these values.

As an alternative to Eqs. (2), one might let

$$u_1 \triangleq \dot{q}_1 c_3 + \dot{q}_2 s_3 \quad u_2 \triangleq -\dot{q}_1 s_3 + \dot{q}_2 c_3 \quad u_3 \triangleq \dot{q}_3 \quad (7)$$

where  $s_3$  and  $c_3$  stand for  $\sin q_3$  and  $\cos q_3$ , respectively. Then  $Z_r = 0$  ( $r = 1, 2, 3$ ), and  $\dot{q}_1, \dot{q}_2, \dot{q}_3$  are given by

$$\dot{q}_1 = u_1 c_3 - u_2 s_3 \quad \dot{q}_2 = u_1 s_3 + u_2 c_3 \quad \dot{q}_3 = u_3 \quad (8)$$

Here, no value of  $\omega t$  needs to be excluded. Finally, suppose that  $u_1, u_2, u_3$  are defined as

$$u_1 \triangleq \dot{q}_1 \quad u_2 \triangleq \dot{q}_2 \quad u_3 \triangleq \dot{q}_3 \quad (9)$$

While these definitions of  $u_1, u_2, u_3$  are simpler than those in Eqs. (2) and Eqs. (7), the latter are preferable in certain contexts, as will presently become apparent.

The motivation for introducing  $u_1, u_2$ , and  $u_3$  as in Eqs. (2), (7), and (9) is the following. The velocity of  $P_1$  in  $A$  can be expressed in a variety of ways, such as

$$\mathbf{v}^{P_1} \underset{(2.6.7)}{=} \dot{q}_1 \mathbf{b}_x + \dot{q}_2 \mathbf{b}_y - \omega q_1 \mathbf{b}_z \quad (10)$$

$$\mathbf{v}^{P_1} \underset{(2.6.8)}{=} (\dot{q}_1 c_3 + \dot{q}_2 s_3) \mathbf{e}_x + (-\dot{q}_1 s_3 + \dot{q}_2 c_3) \mathbf{e}_y - \omega q_1 \mathbf{e}_z \quad (11)$$

and (see Fig. 2.9.2 for  $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ )

$$\mathbf{v}^{P_1} = (\dot{q}_1 \cos \omega t - \omega q_1 \sin \omega t) \mathbf{a}_x + \dot{q}_2 \mathbf{a}_y - (\dot{q}_1 \sin \omega t + \omega q_1 \cos \omega t) \mathbf{a}_z \quad (12)$$

When Eqs. (9) are used to define  $u_1, u_2$ , and  $u_3$ , Eq. (10) can be rewritten as

$$\mathbf{v}^{P_1} = u_1 \mathbf{b}_x + u_2 \mathbf{b}_y - \omega q_1 \mathbf{b}_z \quad (13)$$

Similarly, the definitions of  $u_1, u_2$ , and  $u_3$  in accordance with Eqs. (7) permit one to replace Eq. (11) with a relationship having the same simple form as Eq. (13), namely,

$$\mathbf{v}^{P_1} = u_1 \mathbf{e}_x + u_2 \mathbf{e}_y - \omega q_1 \mathbf{e}_z \quad (14)$$

Finally, with the aid of Eqs. (2), (6), and (12), one obtains

$${}^A\mathbf{v}^{P_1} = u_1 \mathbf{a}_x + u_2 \mathbf{a}_y - (u_1 \tan \omega t + \omega q_1 \sec \omega t) \mathbf{a}_z \quad (15)$$

Here, the third component is a bit more complicated than in Eqs. (13) and (14), but the introduction of generalized speeds has led to a noticeable simplification, nevertheless. The guiding idea in writing Eqs. (7) and (2) was thus to enable one to replace Eqs. (11) and (12), respectively, with expressions having, as nearly as possible, the same simple form as Eq. (10). As for Eq. (9), their use does not lead to any simplifications since Eq. (10) cannot be simplified further, but they were included to show that the concept of generalized speeds remains applicable even under these circumstances.

The simplification of an angular velocity expression through the use of generalized speeds can be illustrated by returning to the example in Sec. 2.4. The angular velocity expression recorded in Eq. (2.4.10), while simple in form, is unsuitable for certain purposes because  $\mathbf{k}_2$ ,  $\mathbf{k}_7$ , and  $\mathbf{k}_3$  are not mutually perpendicular. To overcome this difficulty, one can let  $\mathbf{k}_4$  be a unit vector directed as shown in Fig. 2.4.1 and note that  $\mathbf{k}_3$  then is given by

$$\mathbf{k}_3 = -(\cos q_2 \mathbf{k}_2 + \sin q_2 \mathbf{k}_4) \quad (16)$$

so that Eq. (2.4.10) may be replaced with

$$\begin{aligned} {}^A\boldsymbol{\omega}^B &= \dot{q}_1 \mathbf{k}_2 + \dot{q}_2 \mathbf{k}_7 - \dot{q}_3 (\cos q_2 \mathbf{k}_2 + \sin q_2 \mathbf{k}_4) \\ &= (\dot{q}_1 - \dot{q}_3 \cos q_2) \mathbf{k}_2 + \dot{q}_2 \mathbf{k}_7 - \dot{q}_3 \sin q_2 \mathbf{k}_4 \end{aligned} \quad (17)$$

which reduces to

$${}^A\boldsymbol{\omega}^B = u_1 \mathbf{k}_2 + u_2 \mathbf{k}_7 + u_3 \mathbf{k}_4 \quad (18)$$

if generalized speeds  $u_1$ ,  $u_2$ , and  $u_3$  are defined as

$$u_1 \triangleq \dot{q}_1 - \dot{q}_3 \cos q_2 \quad u_2 \triangleq \dot{q}_2 \quad u_3 \triangleq -\dot{q}_3 \sin q_2 \quad (19)$$

While generalized speeds can be time-derivatives of a function of the generalized coordinates and the time  $t$  [for example,  $u_1$ ,  $u_2$ , and  $u_3$  as defined in Eqs. (2) are, respectively, the time-derivatives of  $q_1 \cos \omega t$ ,  $q_2$ , and  $q_3$ ], this is not always the case. Consider, for instance,  $u_1$  as defined in Eq. (7), and assume the existence of a function  $f$  of  $q_1$ ,  $q_2$ ,  $q_3$ , and  $t$  such that

$$\frac{df}{dt} = u_1 \quad (20)$$

for all values of  $q_1$ ,  $q_2$ ,  $q_3$ , and  $t$  in some domain of these variables. Then

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial q_1} \dot{q}_1 + \frac{\partial f}{\partial q_2} \dot{q}_2 + \frac{\partial f}{\partial q_3} \dot{q}_3 + \frac{\partial f}{\partial t} \\ &= \dot{q}_1 c_3 + \dot{q}_2 s_3 \end{aligned} \quad (21)$$

(7,20)

which implies that

$$\frac{\partial f}{\partial q_1} = c_3 \quad \frac{\partial f}{\partial q_2} = s_3 \quad \frac{\partial f}{\partial q_3} = 0 \quad \frac{\partial f}{\partial t} = 0 \quad (22)$$

The first and third of these equations are incompatible with each other, for they lead to different expressions for  $\partial^2 f / \partial q_1 \partial q_3$ . Thus, the hypothesis that  $f$  exists such that Eq. (20) is satisfied is untenable.

## 2.13 MOTION CONSTRAINTS

It can occur that, for physical reasons, the generalized speeds  $u_1, \dots, u_n$  for a system  $S$  in a reference frame  $A$  (see Sec. 2.4) are not independent of each other. In that event,  $S$  is said to be subject to *motion constraints*, and an equation that relates  $u_1, \dots, u_n$  to each other is called a *nonholonomic constraint equation*.

When a system  $S$  is not subject to motion constraints, then  $S$  is said to be a *holonomic system possessing  $n$  degrees of freedom* in  $A$ . If  $S$  is subject to motion constraints,  $S$  is called a *nonholonomic system*.

When all nonholonomic constraint equations can be expressed as the  $m$  relationships

$$u_r = \sum_{s=1}^p A_{rs} u_s + B_r \quad (r = p + 1, \dots, n) \quad (1)$$

where

$$p \triangleq n - m \quad (2)$$

and where  $A_{rs}$  and  $B_r$  are functions of  $q_1, \dots, q_n$ , and the time  $t$ ,  $S$  is referred to as a *simple nonholonomic system possessing  $p$  degrees of freedom* in  $A$ .

**Example** The particles  $P_1$  and  $P_2$  considered in the example in Sec. 2.9 form a holonomic system possessing three degrees of freedom in  $A$ . Suppose that  $P_2$  is replaced with a small sharp-edged circular disk  $D$  whose axis is normal to the rod  $R$  and parallel to the plane in which  $R$  moves, as indicated in Fig. 2.13.1; further, that  $D$  comes into contact with the two panes of glass at the points  $D_1$  and  $D_2$ . Then it may be assumed that  $D^*$ , the center of  $D$ , can move freely in  $B$  in the direction of  $R$ , but is prevented from moving perpendicularly to  $R$ , a condition that may be stated analytically in terms of  ${}^B\mathbf{v}^{D^*}$ , the velocity of  $D^*$  in  $B$ , as

$${}^B\mathbf{v}^{D^*} \cdot \mathbf{e}_y = 0 \quad (3)$$

$${}^B\mathbf{v}^{D^*} \cdot \mathbf{e}_z = 0 \quad (4)$$

where  $\mathbf{e}_y$  and  $\mathbf{e}_z$  are unit vectors directed as shown in Fig. 2.13.1. Now,

$${}^B\mathbf{v}^{D^*} = {}^A\mathbf{v}^{D^*} - {}^A\mathbf{v}^B \quad (5)$$

(2.8.1)



and Eq. (3) leads to

$$-\dot{q}_1 s_3 + \dot{q}_2 c_3 + L\dot{q}_3 = 0 \quad (9)$$

(3,8)

while Eq. (4) is satisfied identically. Furthermore, if generalized speeds  $u_1$ ,  $u_2$ , and  $u_3$  are introduced as in Eqs. (2.12.2), so that Eqs. (2.12.6) apply, then Eq. (9) gives rise to the nonholonomic constraint equation

$$u_3 = \frac{1}{L}[(u_1 \sec \omega t + \omega q_1 \tan \omega t)s_3 - u_2 c_3] \quad (10)$$

Thus,  $m = 1$ ,  $p = n - m = 3 - 1 = 2$ , and  $P_1$  and  $D$  are seen to form a simple nonholonomic system possessing two degrees of freedom in  $A$ . The functions  $A_{rs}$  and  $B_r$  ( $r = 3$ ;  $s = 1, 2$ ) of Eq. (1) are

$$A_{31} = \frac{s_3}{L} \sec \omega t \quad A_{32} = \frac{-c_3}{L} \quad B_3 = \frac{s_3}{L} \omega q_1 \tan \omega t \quad (11)$$

If Eqs. (2.12.7), rather than Eqs. (2.12.2), are used to define  $u_1$ ,  $u_2$ , and  $u_3$ , then Eq. (10) gives way to the much simpler relationship

$$u_3 = \frac{-u_2}{L} \quad (9) \quad (12)$$

and if  $u_1$ ,  $u_2$ , and  $u_3$  are defined as in Eqs. (2.12.9), the nonholonomic constraint equation is

$$u_3 = \frac{1}{L}(u_1 s_3 - u_2 c_3) \quad (13)$$

(9)

Before leaving this example, it is worth noting that Eq. (9) is nonintegrable; that is, there exists no function  $f(q_1, q_2, q_3)$  which is constant throughout every time interval in which Eq. (9) is satisfied. If such a function existed, then  $q_1$ ,  $q_2$ , and  $q_3$  would not be independent of each other, and thus would not be generalized coordinates.

## 2.14 PARTIAL ANGULAR VELOCITIES, PARTIAL VELOCITIES

If  $q_1, \dots, q_n$  and  $u_1, \dots, u_n$  are, respectively, generalized coordinates (see Sec. 2.10) and generalized speeds (see Sec. 2.12) for a simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a reference frame  $A$ , then  $\omega$ , the angular velocity in  $A$  of a rigid body  $B$  belonging to  $S$ , and  $\mathbf{v}$ , the velocity in  $A$  of a particle  $P$  belonging to  $S$ , can be expressed *uniquely* as

$$\omega = \sum_{r=1}^n \omega_r u_r + \omega_t \quad (1)$$

and

$$\mathbf{v} = \sum_{r=1}^n \mathbf{v}_r u_r + \mathbf{v}_t \quad (2)$$

where  $\omega_r, \mathbf{v}_r$  ( $r = 1, \dots, n$ ),  $\omega_t$ , and  $\mathbf{v}_t$  are functions of  $q_1, \dots, q_n$ , and the time  $t$ . The vector  $\omega_r$  is called the  $r$ th *holonomic partial angular velocity* of  $B$  in  $A$ , and  $\mathbf{v}_r$  is referred to as the  $r$ th *holonomic partial velocity* of  $P$  in  $A$ .

The vectors  $\omega$  and  $\mathbf{v}$  can also be expressed *uniquely* as

$$\omega = \sum_{r=1}^p \tilde{\omega}_r u_r + \tilde{\omega}_t \quad (3)$$

and

$$\mathbf{v} = \sum_{r=1}^p \tilde{\mathbf{v}}_r u_r + \tilde{\mathbf{v}}_t \quad (4)$$

where  $\tilde{\omega}_r, \tilde{\mathbf{v}}_r$  ( $r = 1, \dots, p$ ),  $\tilde{\omega}_t$ , and  $\tilde{\mathbf{v}}_t$  are functions of  $q_1, \dots, q_n$ , and  $t$ . The vector  $\tilde{\omega}_r$  is called the  $r$ th *nonholonomic partial angular velocity* of  $B$  in  $A$ , while  $\tilde{\mathbf{v}}_r$  is known as the  $r$ th *nonholonomic partial velocity* of  $P$  in  $A$ .

When speaking of partial angular velocities and/or partial velocities, one can generally omit the adjectives “holonomic” and “nonholonomic” without loss of clarity, but the tilde notation should be used to distinguish nonholonomic partial angular velocities from holonomic ones, and similarly for partial velocities. When  $p = n$ , that is, when  $S$  is a holonomic system possessing  $n$  degrees of freedom in  $A$ , then  $\tilde{\omega}_r = \omega_r$  and  $\tilde{\mathbf{v}}_r = \mathbf{v}_r$  ( $r = 1, \dots, n$ ). It is customary not to write any tildes under these circumstances.

**Derivation** Solution of Eqs. (2.12.1) for  $\dot{q}_1, \dots, \dot{q}_n$  yields

$$\dot{q}_s = \sum_{r=1}^n W_{sr} u_r + X_s \quad (s = 1, \dots, n) \quad (5)$$

where  $W_{sr}$  and  $X_s$  are certain functions of  $q_1, \dots, q_n$ , and  $t$ . Now, if  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a right-handed set of mutually perpendicular unit vectors fixed in  $B$ , and if  $\dot{\mathbf{b}}_i$  denotes the first time-derivative of  $\mathbf{b}_i$  in  $A$ , then

$$\begin{aligned} \dot{\mathbf{b}}_i &= \sum_{s=1}^n \frac{\partial \mathbf{b}_i}{\partial q_s} \dot{q}_s + \frac{\partial \mathbf{b}_i}{\partial t} \\ &= \sum_{s=1}^n \frac{\partial \mathbf{b}_i}{\partial q_s} \left( \sum_{r=1}^n W_{sr} u_r + X_s \right) + \frac{\partial \mathbf{b}_i}{\partial t} \\ &= \sum_{r=1}^n \sum_{s=1}^n \frac{\partial \mathbf{b}_i}{\partial q_s} W_{sr} u_r + \sum_{s=1}^n \frac{\partial \mathbf{b}_i}{\partial q_s} X_s + \frac{\partial \mathbf{b}_i}{\partial t} \quad (i = 1, 2, 3) \end{aligned} \quad (6)$$

where all partial differentiations of  $\mathbf{b}_i$  are performed in  $A$ . Consequently,

$$\begin{aligned} \omega &= \mathbf{b}_1 \dot{\mathbf{b}}_2 \cdot \mathbf{b}_3 + \mathbf{b}_2 \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 + \mathbf{b}_3 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \\ &= \mathbf{b}_1 \left( \sum_{r=1}^n \sum_{s=1}^n \frac{\partial \mathbf{b}_2}{\partial q_s} \cdot \mathbf{b}_3 W_{sr} u_r + \sum_{s=1}^n \frac{\partial \mathbf{b}_2}{\partial q_s} \cdot \mathbf{b}_3 X_s + \frac{\partial \mathbf{b}_2}{\partial t} \cdot \mathbf{b}_3 \right) \\ &\quad + \mathbf{b}_2 \left( \sum_{r=1}^n \sum_{s=1}^n \frac{\partial \mathbf{b}_3}{\partial q_s} \cdot \mathbf{b}_1 W_{sr} u_r + \sum_{s=1}^n \frac{\partial \mathbf{b}_3}{\partial q_s} \cdot \mathbf{b}_1 X_s + \frac{\partial \mathbf{b}_3}{\partial t} \cdot \mathbf{b}_1 \right) \\ &\quad + \mathbf{b}_3 \left( \sum_{r=1}^n \sum_{s=1}^n \frac{\partial \mathbf{b}_1}{\partial q_s} \cdot \mathbf{b}_2 W_{sr} u_r + \sum_{s=1}^n \frac{\partial \mathbf{b}_1}{\partial q_s} \cdot \mathbf{b}_2 X_s + \frac{\partial \mathbf{b}_1}{\partial t} \cdot \mathbf{b}_2 \right) \end{aligned} \quad (7)$$

and, if  $\omega_r$  and  $\omega_t$  are defined as

$$\omega_r \triangleq \sum_{s=1}^n \left( \mathbf{b}_1 \frac{\partial \mathbf{b}_2}{\partial q_s} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{\partial \mathbf{b}_3}{\partial q_s} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{\partial \mathbf{b}_1}{\partial q_s} \cdot \mathbf{b}_2 \right) W_{sr} \quad (r = 1, \dots, n) \quad (8)$$

and

$$\begin{aligned} \omega_t \triangleq \sum_{s=1}^n \left[ \left( \mathbf{b}_1 \frac{\partial \mathbf{b}_2}{\partial q_s} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{\partial \mathbf{b}_3}{\partial q_s} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{\partial \mathbf{b}_1}{\partial q_s} \cdot \mathbf{b}_2 \right) X_s \right. \\ \left. + \mathbf{b}_1 \frac{\partial \mathbf{b}_2}{\partial q_s} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{\partial \mathbf{b}_3}{\partial q_s} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{\partial \mathbf{b}_1}{\partial q_s} \cdot \mathbf{b}_2 \right] \end{aligned} \quad (9)$$

respectively, then substitution from Eqs. (8) and (9) into Eq. (7) leads directly to Eq. (1).

To establish the validity of Eq. (2), let  $\mathbf{p}$  be the position vector from a point fixed in  $A$  to  $P$ , and let  $\dot{\mathbf{p}}$  denote the first time-derivative of  $P$  in  $A$ . Then

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{p}} = \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} \dot{q}_s + \frac{\partial \mathbf{p}}{\partial t} \\ &\stackrel{(5)}{=} \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} \left( \sum_{r=1}^n W_{sr} u_r + X_s \right) + \frac{\partial \mathbf{p}}{\partial t} \\ &= \sum_{r=1}^n \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sr} u_r + \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} X_s + \frac{\partial \mathbf{p}}{\partial t} \end{aligned} \quad (10)$$

and, after defining  $\mathbf{v}_r$  and  $\mathbf{v}_t$  as

$$\mathbf{v}_r \triangleq \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sr} \quad (r = 1, \dots, n) \quad (11)$$

and

$$\mathbf{v}_t \triangleq \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} X_s + \frac{\partial \mathbf{p}}{\partial t} \quad (12)$$

respectively, one obtains Eq. (2) by substituting from Eqs. (11) and (12) into Eq. (10).

Suppose now that  $S$  is subject to motion constraints such that  $u_1, \dots, u_n$  are governed by Eqs. (2.13.1). Then, after rewriting Eq. (1) as

$$\boldsymbol{\omega} = \sum_{r=1}^p \omega_r u_r + \sum_{r=p+1}^n \omega_r u_r + \boldsymbol{\omega}_t \quad (13)$$

one can use Eq. (2.13.1) to obtain

$$\begin{aligned}
 \omega &= \sum_{r=1}^p \omega_r u_r + \sum_{r=p+1}^n \omega_r \left( \sum_{s=1}^p A_{rs} u_s + B_r \right) + \omega_t \\
 &\quad (13) \quad (2.13.1) \\
 &= \sum_{r=1}^p \omega_r u_r + \sum_{s=1}^p \sum_{r=p+1}^n \omega_r A_{rs} u_s + \sum_{r=p+1}^n \omega_r B_r + \omega_t \\
 &= \sum_{r=1}^p \omega_r u_r + \sum_{r=1}^p \sum_{s=p+1}^n \omega_s A_{sr} u_r + \sum_{r=p+1}^n \omega_r B_r + \omega_t \\
 &= \sum_{r=1}^p \left( \omega_r + \sum_{s=p+1}^n \omega_s A_{sr} \right) u_r + \sum_{r=p+1}^n \omega_r B_r + \omega_t \quad (14)
 \end{aligned}$$

and, after defining  $\tilde{\omega}_r$  and  $\tilde{\omega}_t$  as

$$\tilde{\omega}_r \triangleq \omega_r + \sum_{s=p+1}^n \omega_s A_{sr} \quad (r = 1, \dots, p) \quad (15)$$

and

$$\tilde{\omega}_t \triangleq \omega_t + \sum_{r=p+1}^n \omega_r B_r \quad (16)$$

one arrives at Eq. (3) by substituting from Eqs. (15) and (16) into Eq. (14). A completely analogous derivation leads from Eq. (2) to Eq. (4), provided that  $\tilde{v}_r$  and  $\tilde{v}_t$  are defined as

$$\tilde{v}_r \triangleq v_r + \sum_{s=p+1}^n v_s A_{sr} \quad (r = 1, \dots, p) \quad (17)$$

and

$$\tilde{v}_t \triangleq v_t + \sum_{r=p+1}^n v_r B_r \quad (18)$$

As will become evident later, the use of partial angular velocities and partial velocities greatly facilitates the formulation of equations of motion. Moreover, the constructing of expressions for these quantities is a simple matter involving nothing more than the *inspecting of expressions* for angular velocities of rigid bodies and/or expressions for velocities of particles.

**Example** In the example in Sec. 2.12, three sets of generalized speeds were introduced and the corresponding expressions for the velocity of the particle  $P_1$  in reference frame  $A$  (see Fig. 2.9.1) were recorded in Eqs. (2.12.13)–(2.12.15). Each of these equations has precisely the same form as Eq. (2), and



inspection of the equations thus permits one to identify the associated holonomic partial velocities of  $P_1$  in  $A$  as

$$\begin{matrix} {}^A\mathbf{v}_1^{P_1} = \mathbf{b}_x \\ (2.12.13) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_2^{P_1} = \mathbf{b}_y \\ (2.12.13) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_3^{P_1} = 0 \\ (2.12.13) \end{matrix} \quad (19)$$

$$\begin{matrix} {}^A\mathbf{v}_1^{P_1} = \mathbf{e}_x \\ (2.12.14) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_2^{P_1} = \mathbf{e}_y \\ (2.12.14) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_3^{P_1} = 0 \\ (2.12.14) \end{matrix} \quad (20)$$

$$\begin{matrix} {}^A\mathbf{v}_1^{P_1} = \mathbf{a}_x - \tan \omega t \mathbf{a}_z \\ (2.12.15) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_2^{P_1} = \mathbf{a}_y \\ (2.12.15) \end{matrix} \quad \begin{matrix} {}^A\mathbf{v}_3^{P_1} = 0 \\ (2.12.15) \end{matrix} \quad (21)$$

The angular velocity of  $E$  in  $A$ , introduced in the example in Sec. 2.6, can be written

$${}^A\boldsymbol{\omega}^E = \omega s_3 \mathbf{e}_x + \omega c_3 \mathbf{e}_y + u_3 \mathbf{e}_z \quad (22)$$

(2.6.11)

in all three cases because  $u_3$  was defined as  $\dot{q}_3$  in Eqs. (2.12.2), (2.12.7), and (2.12.9). Comparing Eq. (22) with Eq. (1), one can write down the holonomic partial angular velocities of  $E$  in  $A$ ,

$${}^A\omega_1^E = {}^A\omega_2^E = 0 \quad {}^A\omega_3^E = \mathbf{e}_z \quad (23)$$

To illustrate the idea of nonholonomic partial angular velocities and nonholonomic partial velocities, we confine our attention to the generalized speeds of Eqs. (2.12.7), which means that Eqs. (20) and (23) apply, and explore the effect of the motion constraint considered in the example in Sec. 2.13, which was there shown to give rise to the nonholonomic constraint equation

$$u_3 = -\frac{u_2}{L} \quad (24)$$

(2.13.12)

As regards the partial velocities of  $P_1$  in  $A$ , Eq. (24) makes no difference whatsoever, for  $u_3$  is absent from Eq. (2.12.14), the relevant expression for  ${}^A\mathbf{v}^{P_1}$ . In other words, the two nonholonomic partial velocities of  $P_1$  in  $A$  (there are two because  $n = 3$ ,  $m = 1$ , and  $p = n - m = 3 - 1 = 2$ ) are

$$\begin{matrix} {}^A\tilde{\mathbf{v}}_1^{P_1} = \mathbf{e}_x \\ (2.12.14) \end{matrix} \quad \begin{matrix} {}^A\tilde{\mathbf{v}}_2^{P_1} = \mathbf{e}_y \\ (2.12.14) \end{matrix} \quad (25)$$

and these are the same as their holonomic counterparts in Eqs. (20). In connection with the partial angular velocities of  $E$  in  $A$ , however, Eq. (24) matters very much, for substitution from Eq. (24) into Eq. (22) produces

$${}^A\boldsymbol{\omega}^E = \omega s_3 \mathbf{e}_x + \omega c_3 \mathbf{e}_y - \left(\frac{u_2}{L}\right) \mathbf{e}_z \quad (26)$$

which has the form of Eq. (3) and permits one to identify the two nonholonomic partial angular velocities of  $E$  in  $A$  as

$$\begin{matrix} {}^A\tilde{\omega}_1^E = 0 \\ (26) \end{matrix} \quad \begin{matrix} {}^A\tilde{\omega}_2^E = -\frac{1}{L} \mathbf{e}_z \\ (26) \end{matrix} \quad (27)$$

The second of these differs noticeably from its counterpart in Eq. (23).

Finally, still working with the generalized speeds of Eqs. (2.12.7), let us examine the three holonomic and two nonholonomic partial velocities of  $D^*$  in  $A$  (see Fig. 2.13.1). To determine these, we refer to Eqs. (2.6.13) and (2.12.7) to express  ${}^A\mathbf{v}^{D^*}$  as

$${}^A\mathbf{v}^{D^*} = u_1\mathbf{e}_x + (u_2 + Lu_3)\mathbf{e}_y - \omega(q_1 + Lc_3)\mathbf{e}_z \quad (28)$$

and note that, when Eq. (24) is taken into account,  ${}^A\mathbf{v}^{D^*}$  is given by

$${}^A\mathbf{v}^{D^*} = u_1\mathbf{e}_x - \omega(q_1 + Lc_3)\mathbf{e}_z \quad (29)$$

Consequently, the holonomic partial velocities of  $D^*$  in  $A$  are

$$\begin{array}{lll} {}^A\mathbf{v}_1^{D^*} = \mathbf{e}_x & {}^A\mathbf{v}_2^{D^*} = \mathbf{e}_y & {}^A\mathbf{v}_3^{D^*} = L\mathbf{e}_y \end{array} \quad \begin{array}{l} (28) \\ (28) \\ (28) \end{array} \quad (30)$$

while the nonholonomic partial velocities of  $D^*$  in  $A$  are

$$\begin{array}{ll} {}^A\tilde{\mathbf{v}}_1^{D^*} = \mathbf{e}_x & {}^A\tilde{\mathbf{v}}_2^{D^*} = 0 \end{array} \quad \begin{array}{l} (29) \\ (29) \end{array} \quad (31)$$

## 2.15 ACCELERATION AND PARTIAL VELOCITIES

When  $q_1, \dots, q_n$  are generalized coordinates characterizing the configuration of a system  $S$  in a reference frame  $A$  (see Sec. 2.10), then  $\mathbf{v}^2$ , the square of the velocity in  $A$  of a generic particle  $P$  of  $S$ , may be regarded as a (scalar) function of the  $2n + 1$  independent variables  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , where  $\dot{q}_r$  denotes the first time-derivative of  $q_r$  ( $r = 1, \dots, n$ ). If generalized speeds (see Sec. 2.12) are defined as

$$u_r \triangleq \dot{q}_r \quad (r = 1, \dots, n) \quad (1)$$

and  $\mathbf{v}_r$  denotes the  $r$ th holonomic partial velocity of  $P$  in  $A$  (see Sec. 2.14), then  $\mathbf{v}_r$ , the acceleration  $\mathbf{a}$  of  $P$  in  $A$ , and  $\mathbf{v}^2$  are related to each other as follows:

$$\mathbf{v}_r \cdot \mathbf{a} = \frac{1}{2} \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_r} - \frac{\partial \mathbf{v}^2}{\partial q_r} \right) \quad (r = 1, \dots, n) \quad (2)$$

If, in accordance with Eqs. (2.12.1), generalized speeds are defined as

$$u_r \triangleq \sum_{s=1}^n Y_{rs} \dot{q}_s + Z_r \quad (r = 1, \dots, n) \quad (3)$$

where  $Y_{rs}$  and  $Z_r$  are functions of  $q_1, \dots, q_n$ , and the time  $t$ , and  $\mathbf{v}_r$  denotes the associated  $r$ th holonomic partial velocity of  $P$  in  $A$  (see Sec. 2.14), then

$$\mathbf{v}_r \cdot \mathbf{a} = \frac{1}{2} \sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_s} - \frac{\partial \mathbf{v}^2}{\partial q_s} \right) W_{sr} \quad (r = 1, \dots, n) \quad (4)$$

where  $W_{sr}$  is a function of  $q_1, \dots, q_n$ , and  $t$  such that solution of Eqs. (3) for  $\dot{q}_1, \dots, \dot{q}_n$  yields

$$\dot{q}_s = \sum_{r=1}^n W_{sr} u_r + X_s \quad (s = 1, \dots, n) \quad (5)$$

Finally, when  $S$  is a simple nonholonomic system possessing  $p$  degrees of freedom in  $A$  (see Sec. 2.13), so that there exist  $m$  nonholonomic constraint equations of the form

$$u_r = \sum_{s=1}^p A_{rs} u_s + B_r \quad (r = p+1, \dots, n) \quad (6)$$

then  $\tilde{\mathbf{v}}_r \cdot \mathbf{a}$ , where  $\tilde{\mathbf{v}}_r$  is the  $r$ th nonholonomic partial velocity of  $P$  in  $A$  (see Sec. 2.14), can be expressed in terms of  $\mathbf{v}^2$  (still regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ ) as

$$\begin{aligned} \tilde{\mathbf{v}}_r \cdot \mathbf{a} = & \frac{1}{2} \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_r} - \frac{\partial \mathbf{v}^2}{\partial q_r} \right) \\ & + \frac{1}{2} \sum_{s=p+1}^n \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_s} - \frac{\partial \mathbf{v}^2}{\partial q_s} \right) A_{sr} \quad (r = 1, \dots, p) \end{aligned} \quad (7)$$

when  $u_r$  is defined as in Eqs. (1), and as

$$\tilde{\mathbf{v}}_r \cdot \mathbf{a} = \frac{1}{2} \sum_{s=1}^n \left[ \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_s} - \frac{\partial \mathbf{v}^2}{\partial q_s} \right) \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \right] \quad (r = 1, \dots, p) \quad (8)$$

when  $u_r$  is defined as in Eq. (3).

Equations (2), (7), and (8) play essential parts in the derivations of Lagrange equations and Passerello-Huston equations (see Problem 11.12). Additionally, Eqs. (2) can facilitate the determination of accelerations, as will be shown presently.

**Derivation** When  $u_r$  is defined as in Eqs. (1), then  $W_{sr}$  in Eqs. (2.14.5) is equal to unity for  $s = r$  and vanishes otherwise, while  $X_s$  vanishes for  $s = 1, \dots, n$ . Consequently, Eqs. (2.14.11) and (2.14.12) reduce to

$$\mathbf{v}_r = \frac{\partial \mathbf{p}}{\partial q_r} \quad (r = 1, \dots, n) \quad (9)$$

$$\mathbf{v}_t = \frac{\partial \mathbf{p}}{\partial t} \quad (10)$$

respectively. From the first of these it follows that

$$\frac{\partial \mathbf{v}_r}{\partial q_s} = \frac{\partial}{\partial q_s} \left( \frac{\partial \mathbf{p}}{\partial q_r} \right) = \frac{\partial}{\partial q_r} \left( \frac{\partial \mathbf{p}}{\partial q_s} \right) = \frac{\partial \mathbf{v}_s}{\partial q_r} \quad (r, s = 1, \dots, n) \quad (11)$$

while the two together lead to

$$\frac{\partial \mathbf{v}_t}{\partial q_r} \underset{(10)}{=} \frac{\partial}{\partial q_r} \left( \frac{\partial \mathbf{p}}{\partial t} \right) \underset{(1.8.4)}{=} \frac{\partial}{\partial t} \frac{\partial \mathbf{p}}{\partial q_r} \underset{(9)}{=} \frac{\partial \mathbf{v}_r}{\partial t} \quad (12)$$

These relationships will be used shortly.

When  $\mathbf{v}$  is expressed as in Eq. (2.14.2) and  $u_r$  is replaced with  $\dot{q}_r$  in accordance with Eqs. (1), one can regard  $\mathbf{v}$  as a function of the independent variables  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , in which event

$$\frac{\partial \dot{q}_s}{\partial q_r} = 0 \quad (r, s = 1, \dots, n) \quad (13)$$

and partial differentiation of  $\mathbf{v}$  with respect to  $q_r$  gives

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial q_r} &= \frac{\partial}{\partial q_r} \left( \sum_{s=1}^n \mathbf{v}_s \dot{q}_s + \mathbf{v}_t \right) \underset{(1)}{=} \sum_{s=1}^n \left( \frac{\partial \mathbf{v}_s}{\partial q_r} \dot{q}_s + \mathbf{v}_s \frac{\partial \dot{q}_s}{\partial q_r} \right) + \frac{\partial \mathbf{v}_t}{\partial q_r} \\ &\underset{(1.7.1, 1.7.2)}{=} \sum_{s=1}^n \frac{\partial \mathbf{v}_r}{\partial q_s} \dot{q}_s + 0 + \frac{\partial \mathbf{v}_t}{\partial q_r} \underset{(13)}{=} \frac{d\mathbf{v}_r}{dt} \underset{(1.9.1)}{=} \frac{d\mathbf{v}_r}{dt} \quad (r = 1, \dots, n) \end{aligned} \quad (14)$$

while partial differentiation with respect to  $\dot{q}_r$  produces

$$\frac{\partial \mathbf{v}}{\partial \dot{q}_r} \underset{(2.14.2)}{=} \frac{\partial}{\partial \dot{q}_r} \left( \sum_{s=1}^n \mathbf{v}_s \dot{q}_s + \mathbf{v}_t \right) = \mathbf{v}_r \quad (r = 1, \dots, n) \quad (15)$$

since  $\mathbf{v}_s$  and  $\mathbf{v}_t$  are independent of  $\dot{q}_r$ , and  $\partial \dot{q}_s / \partial \dot{q}_r$  vanishes except for  $s = r$ , in which case it is equal to unity.

To conclude the derivation of Eq. (2), we note that

$$\begin{aligned} \frac{d}{dt} (\mathbf{v}_r \cdot \mathbf{v}) &= \frac{d\mathbf{v}_r}{dt} \cdot \mathbf{v} + \mathbf{v}_r \cdot \frac{d\mathbf{v}}{dt} \\ &= \frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{v} + \mathbf{v}_r \cdot \mathbf{a} \underset{(2.6.2)}{=} \underset{(14)}{\frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{v}} + \mathbf{v}_r \cdot \mathbf{a} \quad (r = 1, \dots, n) \end{aligned} \quad (16)$$

which, solved for  $\mathbf{v}_r \cdot \mathbf{a}$ , yields

$$\mathbf{v}_r \cdot \mathbf{a} \underset{(16)}{=} \frac{d}{dt} (\mathbf{v}_r \cdot \mathbf{v}) - \frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{v} \quad (r = 1, \dots, n) \quad (17)$$

$$\begin{aligned} \mathbf{v}_r \cdot \mathbf{a} &\underset{(17)}{=} \frac{d}{dt} \left( \frac{\partial \mathbf{v}}{\partial \dot{q}_r} \cdot \mathbf{v} \right) - \frac{\partial \mathbf{v}}{\partial q_r} \cdot \mathbf{v} \\ &= \frac{d}{dt} \left( \frac{1}{2} \frac{\partial v^2}{\partial \dot{q}_r} \right) - \frac{1}{2} \frac{\partial v^2}{\partial q_r} \quad (r = 1, \dots, n) \end{aligned} \quad (18)$$

and this is equivalent to Eqs. (2).

To establish the validity of Eqs. (4), we begin by exploring the relationship between the partial velocities associated with generalized speeds defined as in Eqs. (1), on the one hand, and partial velocities associated with generalized speeds defined as in Eqs. (3), on the other hand. Denoting the former by  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , as heretofore, we have

$$\mathbf{v} = \sum_{r=1}^n \mathbf{v}_r \dot{q}_r + \mathbf{v}_t \quad (19)$$

(2.14.2) (1)

Hence, when  $u_1, \dots, u_n$  are defined as in Eqs. (3), so that Eqs. (5) apply, we can write

$$\begin{aligned} \mathbf{v} &= \sum_{r=1}^n \mathbf{v}_r \left( \sum_{s=1}^n W_{rs} u_s + X_r \right) + \mathbf{v}_t \\ &= \sum_{r=1}^n \sum_{s=1}^n \mathbf{v}_s W_{sr} u_r + \sum_{r=1}^n \mathbf{v}_r X_r + \mathbf{v}_t \end{aligned} \quad (20)$$

and now we can identify the partial velocities associated with  $u_1, \dots, u_n$ , which we denote temporarily by  $\bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_n$ , as the coefficients of  $u_1, \dots, u_n$ , respectively, in Eq. (20); that is,

$$\bar{\mathbf{v}}_r \triangleq \sum_{s=1}^n \mathbf{v}_s W_{sr} \quad (r = 1, \dots, n) \quad (21)$$

The derivation of Eqs. (4) then can be completed by dot-multiplying Eqs. (21) with  $\mathbf{a}$ , using Eqs. (2) to eliminate  $\mathbf{v}_s \cdot \mathbf{a}$  ( $s = 1, \dots, n$ ), and writing  $\mathbf{v}_r$  in place of  $\bar{\mathbf{v}}_r$  ( $r = 1, \dots, n$ ).

Lastly, to obtain Eqs. (7) and (8), dot-multiply Eqs. (2.14.17) with  $\mathbf{a}$ , showing that

$$\bar{\mathbf{v}}_r \cdot \mathbf{a} = \mathbf{v}_r \cdot \mathbf{a} + \sum_{s=p+1}^n \mathbf{v}_s \cdot \mathbf{a} A_{sr} \quad (r = 1, \dots, p) \quad (22)$$

and then use Eqs. (2) in connection with Eqs. (7), and Eqs. (4) in the case of Eqs. (8), to eliminate  $\mathbf{v}_r \cdot \mathbf{a}$  ( $r = 1, \dots, p$ ) and  $\mathbf{v}_s \cdot \mathbf{a}$  ( $s = p+1, \dots, n$ ).

**Example** Considering a point  $P$  moving in a reference frame  $A$ , let  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  be mutually perpendicular unit vectors fixed in  $A$ , and express  $\mathbf{p}$ , the position vector from a point fixed in  $A$  to point  $P$ , as

$$\mathbf{p} = p_1 \mathbf{a}_1 + p_2 \mathbf{a}_2 + p_3 \mathbf{a}_3 \quad (23)$$

where  $p_r$  ( $r = 1, 2, 3$ ) are single-valued functions of three scalar variables,  $q_r$  ( $r = 1, 2, 3$ ). Then there corresponds to every set of values of  $q_r$  ( $r = 1, 2, 3$ ) a unique position of  $P$  in  $A$ ;  $q_r$  ( $r = 1, 2, 3$ ) are called *curvilinear coordinates* of  $P$  in  $A$ ; and the partial derivatives of  $\mathbf{p}$  with respect to  $q_r$  ( $r = 1, 2, 3$ ) in  $A$  can be expressed as

$$\frac{\partial \mathbf{p}}{\partial q_r} = f_r \mathbf{n}_r \quad (r = 1, 2, 3) \quad (24)$$

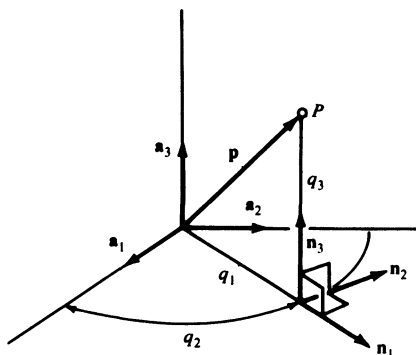


Figure 2.15.1

where  $f_r$  is a function of  $q_1, q_2, q_3$ , and  $\mathbf{n}_r$  is a unit vector. If  $\mathbf{n}_1, \mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular, then  $q_1, q_2$ , and  $q_3$  are called *orthogonal curvilinear coordinates*. For instance, suppose that  $q_1, q_2$ , and  $q_3$  measure two distances and an angle as indicated in Fig. 2.15.1. Then

$$\mathbf{p} = q_1 \cos q_2 \mathbf{a}_1 + q_1 \sin q_2 \mathbf{a}_2 + q_3 \mathbf{a}_3 \quad (25)$$

so that

$$\frac{\partial \mathbf{p}}{\partial q_1} \stackrel{(25)}{=} \cos q_2 \mathbf{a}_1 + \sin q_2 \mathbf{a}_2 \quad (26)$$

$$\frac{\partial \mathbf{p}}{\partial q_2} \stackrel{(25)}{=} -q_1 \sin q_2 \mathbf{a}_1 + q_1 \cos q_2 \mathbf{a}_2 \quad (27)$$

$$\frac{\partial \mathbf{p}}{\partial q_3} \stackrel{(25)}{=} \mathbf{a}_3 \quad (28)$$

and the functions  $f_r$  and unit vectors  $\mathbf{n}_r$  ( $r = 1, 2, 3$ ) appearing in Eqs. (24) can be identified as

$$f_1 = 1 \quad \mathbf{n}_1 = \cos q_2 \mathbf{a}_1 + \sin q_2 \mathbf{a}_2 \quad (29)$$

$$f_2 = q_1 \quad \mathbf{n}_2 = -\sin q_2 \mathbf{a}_1 + \cos q_2 \mathbf{a}_2 \quad (30)$$

$$f_3 = 1 \quad \mathbf{n}_3 = \mathbf{a}_3 \quad (31)$$

Moreover, it follows from Eqs. (29)–(31) that

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{n}_3 = \mathbf{n}_3 \cdot \mathbf{n}_1 = 0 \quad (32)$$

which means that  $\mathbf{n}_1, \mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular, as indicated in Fig. 2.15.1. Consequently,  $q_1, q_2$ , and  $q_3$  are orthogonal curvilinear coordinates of  $P$  in  $A$ .

When  $q_r$  ( $r = 1, 2, 3$ ) are orthogonal curvilinear coordinates of  $P$  in  $A$ , the acceleration  $\mathbf{a}$  of  $P$  in  $A$  can be expressed as

$$\mathbf{a} = a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + a_3 \mathbf{n}_3 \quad (33)$$

where  $\mathbf{n}_r$  ( $r = 1, 2, 3$ ) are the unit vectors appearing in Eqs. (24). To find an expression for  $a_r$  ( $r = 1, 2, 3$ ) in terms of  $f_1, f_2, f_3$  and  $q_1, q_2, q_3$ , one can employ Eqs. (2) after noting that the velocity  $\mathbf{v}$  of  $P$  in  $A$  is given by

$$\mathbf{v} \underset{(2.6.1)}{=} \frac{d\mathbf{p}}{dt} \underset{(1.9.1)}{=} \sum_{r=1}^3 \frac{\partial \mathbf{p}}{\partial q_r} \dot{q}_r \underset{(24)}{=} \sum_{r=1}^3 f_r \dot{q}_r \mathbf{n}_r \quad (34)$$

so that, if  $u_r$  is defined as in Eqs. (1), then

$$\mathbf{v} \underset{(34)}{=} \sum_{r=1}^3 f_r u_r \mathbf{n}_r \quad (35)$$

which means that

$$\mathbf{v}_r \underset{(2.14.2)}{=} f_r \mathbf{n}_r \quad (r = 1, 2, 3) \quad (36)$$

and

$$\mathbf{v}_1 \cdot \mathbf{a} = f_1 \mathbf{n}_1 \cdot (a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + a_3 \mathbf{n}_3) \underset{(33)}{=} a_1 f_1 \quad (37)$$

But

$$\mathbf{v}_1 \cdot \mathbf{a} \underset{(2)}{=} \frac{1}{2} \left( \frac{d}{dt} \frac{\partial \mathbf{v}^2}{\partial \dot{q}_1} - \frac{\partial \mathbf{v}^2}{\partial q_1} \right) \quad (38)$$

and

$$\mathbf{v}^2 \underset{(34)}{=} \sum_{r=1}^3 (f_r \dot{q}_r)^2 \quad (39)$$

so that

$$\frac{\partial \mathbf{v}^2}{\partial \dot{q}_1} \underset{(39)}{=} 2 f_1^2 \dot{q}_1 \quad \frac{\partial \mathbf{v}^2}{\partial q_1} \underset{(39)}{=} 2 \sum_{s=1}^3 f_s \frac{\partial f_s}{\partial q_1} \dot{q}_s^2 \quad (40)$$

and

$$\mathbf{v}_1 \cdot \mathbf{a} \underset{(38,40)}{=} \frac{d}{dt} (f_1^2 \dot{q}_1) - \sum_{s=1}^3 f_s \frac{\partial f_s}{\partial q_1} \dot{q}_s^2 \quad (41)$$

Substituting from Eq. (41) into Eq. (37) and solving for  $a_1$ , one thus finds that  $a_1$  is given by

$$a_1 = \frac{1}{f_1} \left[ \frac{d}{dt} (f_1^2 \dot{q}_1) - \sum_{s=1}^3 f_s \frac{\partial f_s}{\partial q_1} \dot{q}_s^2 \right] \quad (42)$$

and, after using similar processes in connection with  $a_2$  and  $a_3$ , one can conclude that

$$a_r = \frac{1}{f_r} \left[ \frac{d}{dt} (f_r^2 \dot{q}_r) - \sum_{s=1}^3 f_s \frac{\partial f_s}{\partial q_r} \dot{q}_s^2 \right] \quad (r = 1, 2, 3) \quad (43)$$

To illustrate the use of this quite general formula, we return to the orthogonal curvilinear coordinates  $q_1, q_2, q_3$  introduced in Eq. (25). Expressions for  $f_r$  ( $r = 1, 2, 3$ ) available in Eqs. (29)–(31) permit us to write

$$\frac{d}{dt}(f_1^2 \dot{q}_1)_{(29)} = \ddot{q}_1 \quad \frac{\partial f_1}{\partial q_1}_{(29)} = 0 \quad \frac{\partial f_2}{\partial q_1}_{(30)} = 1 \quad \frac{\partial f_3}{\partial q_1}_{(31)} = 0 \quad (44)$$

and, therefore,

$$a_1 = \ddot{q}_1 - q_1 \dot{q}_2^2 \quad (43, 44)$$

Similarly,

$$\frac{d}{dt}(f_2^2 \dot{q}_2)_{(30)} = \frac{d}{dt}(q_1^2 \dot{q}_2) \quad \frac{\partial f_s}{\partial q_2}_{(29-31)} = 0 \quad (s = 1, 2, 3) \quad (46)$$

so that

$$a_2 = \frac{1}{q_1} \frac{d}{dt}(q_1^2 \dot{q}_2) \quad (43, 46)$$

and

$$\frac{d}{dt}(f_3^2 \dot{q}_3)_{(31)} = \ddot{q}_3 \quad \frac{\partial f_s}{\partial q_3}_{(29-31)} = 0 \quad (s = 1, 2, 3) \quad (48)$$

which means that

$$a_3 = \ddot{q}_3 \quad (43, 48)$$

Thus, we now can express the acceleration of  $P$  in  $A$  as

$$\mathbf{a}_{(33)} = (\ddot{q}_1 - q_1 \dot{q}_2^2)_{(45)} \mathbf{n}_1 + \frac{1}{q_1} \frac{d}{dt}(q_1^2 \dot{q}_2)_{(47)} \mathbf{n}_2 + \ddot{q}_3_{(49)} \mathbf{n}_3 \quad (50)$$



## MASS DISTRIBUTION

The motion that results when forces act on a material system depends not only on the forces, but also on the constitution of the system. In particular, the manner in which mass is distributed throughout a system generally affects the behavior of the system. For example, suppose that a rod is supported at one end by a fixed horizontal pin and that a relatively heavy particle is attached at a point of the rod, so that together the rod and the particle form a pendulum. The frequency of the oscillations that ensue when the pendulum is released from rest after having been displaced from the vertical depends on the location of the particle along the rod, that is, on the manner in which mass is distributed throughout the pendulum.

For the purpose of certain analyses, it is unnecessary to know in detail how mass is distributed throughout each of the bodies forming a system; all one needs to know for each body is the location of the mass center, as well as the values of six quantities called inertia scalars. The subject of mass center location is considered in Secs. 3.1 and 3.2. Products of inertia and moments of inertia, which are inertia scalars, are defined in Sec. 3.3 in terms of quantities called inertia vectors. Sections 3.4–3.7 deal with the evaluation of inertia scalars, in which connection inertia matrices and inertia dyadics are discussed. A special kind of moment of inertia, called a principal moment of inertia, is introduced in Sec. 3.8. The chapter concludes with an examination of the relationship between principal moments of inertia, on the one hand, and maximum and minimum moments of inertia, on the other hand.

### 3.1 MASS CENTER

If  $S$  is a set of particles  $P_1, \dots, P_v$  of masses  $m_1, \dots, m_v$ , respectively, there exists a unique point  $S^*$  such that

$$\sum_{i=1}^v m_i \mathbf{r}_i = 0 \quad (1)$$

where  $\mathbf{r}_i$  is the position vector from  $S^*$  to  $P_i$  ( $i = 1, \dots, v$ ).  $S^*$ , called the *mass center* of  $S$ , can be located as follows. Let  $O$  be any point whatsoever, and let  $\mathbf{p}_i$  be the position vector from  $O$  to  $P_i$  ( $i = 1, \dots, v$ ). Then  $\mathbf{p}^*$ , the position vector from  $O$  to  $S^*$ , is given by

$$\mathbf{p}^* = \frac{\sum_{i=1}^v m_i \mathbf{p}_i}{\sum_{i=1}^v m_i} \quad (2)$$

**Derivation** With  $\mathbf{p}_i$  as defined, introduce a point  $\tilde{S}$ , let  $\tilde{\mathbf{p}}$  be the position vector from  $O$  to  $\tilde{S}$ , and let  $\tilde{\mathbf{r}}_i$  be the position vector from  $\tilde{S}$  to  $P_i$  ( $i = 1, \dots, v$ ). Then

$$\tilde{\mathbf{r}}_i = \mathbf{p}_i - \tilde{\mathbf{p}} \quad (i = 1, \dots, v) \quad (3)$$

and

$$\begin{aligned} \sum_{i=1}^v m_i \tilde{\mathbf{r}}_i &= \sum_{(3) i=1}^v m_i \mathbf{p}_i - \sum_{i=1}^v m_i \tilde{\mathbf{p}} \\ &= \sum_{i=1}^v m_i \mathbf{p}_i - \left( \sum_{i=1}^v m_i \right) \tilde{\mathbf{p}} \end{aligned} \quad (4)$$

Set the right-hand member of this equation equal to zero, solve the resulting equation for  $\tilde{\mathbf{p}}$ , and call the value of  $\tilde{\mathbf{p}}$  thus obtained  $\mathbf{p}^*$ . This produces Eq. (2). Next, replace  $\tilde{\mathbf{p}}$  with  $\mathbf{p}^*$  in Eq. (3) and let  $\mathbf{r}_i$  denote the resulting value of  $\tilde{\mathbf{r}}_i$ . Then  $\tilde{\mathbf{r}}_i$  may be replaced with  $\mathbf{r}_i$  in Eq. (4) whenever  $\tilde{\mathbf{p}}$  is replaced with  $\mathbf{p}^*$ , and under these circumstances Eq. (4) reduces to Eq. (1).

**Example** A process called “static balancing” consists of adding matter to, or removing matter from, an object (e.g., an automobile wheel) in such a way as to minimize the distance from the mass center of the new object thus created to a specified line (e.g., the axle of the wheel). Consider, for instance, a set of three particles  $P_1, P_2, P_3$  situated at corners of a cube as shown in Fig. 3.1.1

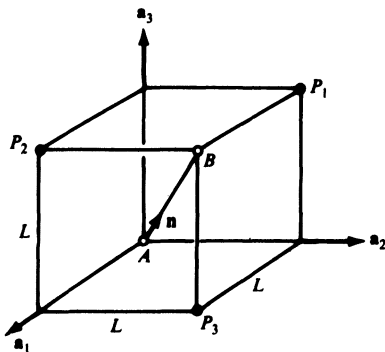


Figure 3.1.1

and having masses  $m$ ,  $2m$ ,  $3m$ , respectively. The mass center of this set of particles does not lie on line  $AB$ , but, by replacing  $P_3$  with a particle  $Q$  of mass  $\mu$  and choosing  $\mu$  suitably, one can minimize the distance from line  $AB$  to the mass center  $S^*$  of the set  $S$  of particles  $P_1$ ,  $P_2$ , and  $Q$ . To determine  $\mu$ , introduce  $\mathbf{p}^*$  as the position vector from  $A$  to the mass center  $S^*$  of  $S$ , let  $\mathbf{n}$  be a unit vector directed as shown in Fig. 3.1.1, and note that  $D^2$ , the square of the distance  $D$  from  $S^*$  to line  $AB$ , is given by

$$D^2 = (\mathbf{p}^* \times \mathbf{n})^2 \quad (5)$$

Now, if  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  are unit vectors directed as in Fig. 3.1.1, then

$$\mathbf{p}^* = \frac{m(\mathbf{a}_2 + \mathbf{a}_3) + 2m(\mathbf{a}_3 + \mathbf{a}_1) + \mu(\mathbf{a}_1 + \mathbf{a}_2)}{3m + \mu} L \quad (6)$$

while

$$\mathbf{n} = \frac{\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3}{\sqrt{3}} \quad (7)$$

Consequently,

$$D^2 = \frac{2L^2(\mu^2 - 3\mu m + 3m^2)}{3(3m + \mu)^2} \quad (8)$$

and, since a value of  $\mu$  that minimizes  $D$  must satisfy the requirement

$$\frac{dD^2}{d\mu} = 0 \quad (9)$$

it may be verified that  $\mu = 5m/3$ . In accordance with Eq. (8), the associated (minimum) distance from line  $AB$  to  $S^*$  is equal to  $L/\sqrt{42}$ .

### 3.2 CURVES, SURFACES, AND SOLIDS

When a body  $B$  is modeled as matter distributed along a curve, over a surface, or throughout a solid, there exists a unique point  $B^*$  such that

$$\int_F \rho \mathbf{r} d\tau = 0 \quad (1)$$

where  $\rho$  is the mass density (i.e., the mass per unit of length, area, or volume) of  $B$  at a generic point  $P$  of  $B$ ,  $\mathbf{r}$  is the position vector from  $B^*$  to  $P$ ,  $d\tau$  is the length, area, or volume of a differential element of the figure  $F$  (curve, surface, or solid) occupied by  $B$ , and the integration is extended throughout  $F$ .  $B^*$ , called the *mass center* of  $B$ , can be located as follows. Let  $O$  be any point whatsoever, and let  $\mathbf{p}$

be the position vector from  $O$  to  $P$ . Then  $\mathbf{p}^*$ , the position vector from  $O$  to  $B^*$ , is given by

$$\mathbf{p}^* = \frac{\int_F \rho \mathbf{p} \, d\tau}{\int_F \rho \, d\tau} \quad (2)$$

The lines of reasoning leading to Eqs. (1) and (2) are analogous to those followed in connection with Eqs. (3.1.1) and (3.1.2).

When  $\rho$  varies from point to point of  $B$ , the integrals appearing in Eq. (2) generally must be worked out by the analyst who wishes to locate  $B^*$ ; but when  $B$  is a *uniform* body, that is, when  $\rho$  is independent of the position of  $P$  in  $B$ , then the desired information frequently is readily available, for  $B^*$  then coincides with the *centroid* of the figure  $F$  occupied by  $B$ ; the centroids of many figures have been found (by integration), and the results have been recorded, as in Appendix I. This information makes it possible to locate, without performing any integrations, the mass center of any body  $B$  that can be regarded as composed solely of uniform bodies  $B_1, \dots, B_v$  whose masses and mass center locations are known. Under these circumstances, the mass center of  $B$  coincides with the mass center of a (fictitious) set of particles (see Sec. 3.1) whose masses are those of  $B_1, \dots, B_v$ , and which are situated at the mass centers of  $B_1, \dots, B_v$ , respectively.

**Example** Figure 3.2.1 shows a body  $B$  consisting of a wire,  $EFG$ , attached to a piece of sheet metal,  $EGH$ . (The lines  $X_1, X_2, X_3$  are mutually perpendicular.) The mass of the sheet metal is 10 times that of the wire.

To locate the mass center  $B^*$  of  $B$ , we let  $B_1$  and  $B_2$  be bodies formed by matter distributed uniformly along the straight line  $EF$  and the circular curve  $FG$ , respectively, and model  $B_3$ , the sheet metal portion of  $B$ , as matter distributed uniformly over the plane triangular surface  $EGH$ . The mass centers of  $B_1, B_2$ , and  $B_3$ , found by reference to Appendix I, are the points  $B_1^*, B_2^*$ , and

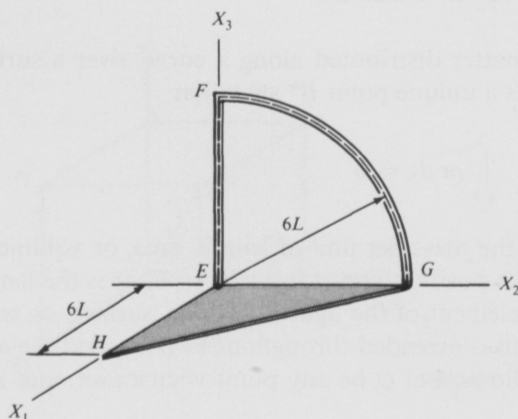


Figure 3.2.1

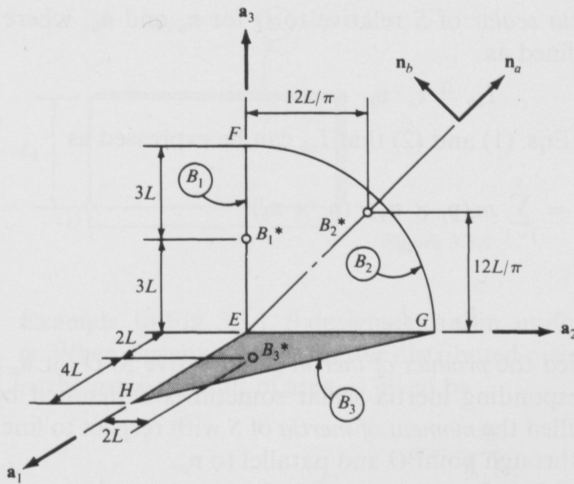


Figure 3.2.2

$B_3^*$  in Fig. 3.2.2, and the masses  $m_1$ ,  $m_2$ , and  $m_3$  of  $B_1$ ,  $B_2$ , and  $B_3$  are taken to be

$$m_1 = m \quad m_2 = \left(\frac{\pi}{2}\right)m \quad m_3 = 10\left(1 + \frac{\pi}{2}\right)m \quad (3)$$

where  $m$  is arbitrary. In accordance with Eq. (3.1.2), the position vector  $\mathbf{p}^*$  from point  $E$  to the mass center of three particles situated at  $B_1^*$ ,  $B_2^*$ , and  $B_3^*$ , and having masses  $m_1$ ,  $m_2$ , and  $m_3$ , respectively, is given by

$$\begin{aligned} \mathbf{p}^* &= \frac{m_1(3L\mathbf{a}_3) + m_2[(12L/\pi)\mathbf{a}_2 + (12L/\pi)\mathbf{a}_3] + m_3(2L\mathbf{a}_1 + 2L\mathbf{a}_2)}{m_1 + m_2 + m_3} \\ &= L \frac{20(1 + \pi/2)\mathbf{a}_1 + [6 + 20(1 + \pi/2)]\mathbf{a}_2 + 9\mathbf{a}_3}{11(1 + \pi/2)} \\ &= (1.82\mathbf{a}_1 + 2.03\mathbf{a}_2 + 0.318\mathbf{a}_3)L \end{aligned} \quad (4)$$

The vector  $\mathbf{p}^*$  is the position vector from  $E$  to  $B^*$ , the mass center of  $B$ , and Eq. (4) shows that  $B^*$  lies neither on the wire nor on the sheet metal portion of  $B$ .

### 3.3 INERTIA VECTOR, INERTIA SCALARS

If  $S$  is a set of particles  $P_1, \dots, P_v$  of masses  $m_1, \dots, m_v$ , respectively,  $\mathbf{p}_i$  is the position vector from a point  $O$  to  $P_i$  ( $i = 1, \dots, v$ ), and  $\mathbf{n}_a$  is a unit vector, then a vector  $\mathbf{I}_a$ , called the *inertia vector* of  $S$  relative to  $O$  for  $\mathbf{n}_a$ , is defined as

$$\mathbf{I}_a \triangleq \sum_{i=1}^v m_i \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \quad (1)$$

A scalar  $I_{ab}$ , called the *inertia scalar* of  $S$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$ , where  $\mathbf{n}_b$ , like  $\mathbf{n}_a$ , is a unit vector, is defined as

$$I_{ab} \triangleq \mathbf{I}_a \cdot \mathbf{n}_b \quad (2)$$

It follows immediately from Eqs. (1) and (2) that  $I_{ab}$  can be expressed as

$$I_{ab} = \sum_{i=1}^v m_i (\mathbf{p}_i \times \mathbf{n}_a) \cdot (\mathbf{p}_i \times \mathbf{n}_b) \quad (3)$$

and this shows that

$$I_{ab} = I_{ba} \quad (4)$$

When  $\mathbf{n}_b \neq \mathbf{n}_a$ ,  $I_{ab}$  is called the *product of inertia* of  $S$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$ . When  $\mathbf{n}_b = \mathbf{n}_a$ , the corresponding inertia scalar sometimes is denoted by  $I_a$  (rather than by  $I_{aa}$ ), and is called the *moment of inertia* of  $S$  with respect to line  $L_a$ , where  $L_a$  is the line passing through point  $O$  and parallel to  $\mathbf{n}_a$ .

The moment of inertia of  $S$  with respect to a line  $L_a$  can always be expressed both as

$$I_a = \sum_{i=1}^v m_i l_i^2 \quad (5)$$

where  $l_i$  is the distance from  $P_i$  to line  $L_a$ , and as

$$I_a = mk_a^2 \quad (6)$$

where  $m$  is the total mass of  $S$ , and  $k_a$  is a real, non-negative quantity called the *radius of gyration* of  $S$  with respect to line  $L_a$ . Equation (5) follows from the fact that

$$I_a = \sum_{(3) i=1}^v m_i (\mathbf{p}_i \times \mathbf{n}_a)^2 \quad (7)$$

and that  $\mathbf{p}_i \times \mathbf{n}_a$  has the magnitude  $l_i$ , the vector  $\mathbf{p}_i$  being the position vector from a point on  $L_a$  to  $P_i$ .

Inertia vectors, products of inertia, moments of inertia, and radii of gyration of a body  $B$  modeled as matter distributed along a curve, over a surface, or throughout a solid are defined analogously. Specifically, if  $\rho$  is the mass density (i.e., the mass per unit of length, area, or volume) of  $B$  at a generic point  $P$  of  $B$ ,  $\mathbf{p}$  is the position vector from a point  $O$  to  $P$ ,  $d\tau$  is the length, area, or volume of a differential element of the figure  $F$  (curve, surface, or solid) occupied by  $B$ , and  $\mathbf{n}_a$  and  $\mathbf{n}_b$  are unit vectors, then Eqs. (1), (2), and (5) give way, respectively, to

$$\mathbf{I}_a \triangleq \int_F \rho \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) d\tau \quad (8)$$

$$I_{ab} \triangleq \mathbf{I}_a \cdot \mathbf{n}_b = \int_F \rho (\mathbf{p} \times \mathbf{n}_a) \cdot (\mathbf{p} \times \mathbf{n}_b) d\tau \quad (9)$$

$$I_a = \int_F \rho l^2 d\tau \quad (10)$$

where  $l$  is the distance from  $P$  to line  $L_a$ ; and Eq. (6) applies to  $B$  as well as to  $S$ .

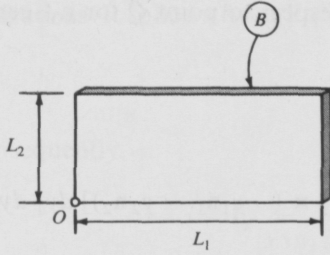


Figure 3.3.1

**Example** In Fig. 3.3.1,  $B$  designates a thin, uniform, rectangular plate of mass  $m$ . When  $B$  is modeled as matter distributed over a rectangular surface  $R$ , then  $\rho$ , the mass per unit of area, is given by

$$\rho = \frac{m}{L_1 L_2} \quad (11)$$

while  $\mathbf{p}$ , the position vector from point  $O$  to a generic point  $P$  of  $B$ , can be expressed as

$$\mathbf{p} = x_1 \mathbf{n}_1 + x_2 \mathbf{n}_2 \quad (12)$$

where  $x_1$  and  $x_2$  are distances, and  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are unit vectors, as shown in Fig. 3.3.2. The inertia vector of  $B$  with respect to  $O$  for  $\mathbf{n}_1$  is thus

$$\begin{aligned} \mathbf{I}_1^O &= \int_R \rho \mathbf{p} \times (\mathbf{n}_1 \times \mathbf{p}) d\tau \\ (8) \quad &= \int_0^{L_2} \int_0^{L_1} \frac{m}{L_1 L_2} (x_1 \mathbf{n}_1 + x_2 \mathbf{n}_2) \times [\mathbf{n}_1 \times (x_1 \mathbf{n}_1 + x_2 \mathbf{n}_2)] dx_1 dx_2 \\ (11, 12) \quad &= \frac{m}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} (x_2^2 \mathbf{n}_1 - x_1 x_2 \mathbf{n}_2) dx_1 dx_2 \\ &= \frac{m}{L_1 L_2} \left( \mathbf{n}_1 \int_0^{L_2} \int_0^{L_1} x_2^2 dx_1 dx_2 - \mathbf{n}_2 \int_0^{L_2} \int_0^{L_1} x_1 x_2 dx_1 dx_2 \right) \\ &= \frac{m}{L_1 L_2} \left( \mathbf{n}_1 \frac{L_1 L_2^3}{3} - \mathbf{n}_2 \frac{L_1^2 L_2^2}{4} \right) = mL_2 \left( \frac{L_2}{3} \mathbf{n}_1 - \frac{L_1}{4} \mathbf{n}_2 \right) \quad (13) \end{aligned}$$

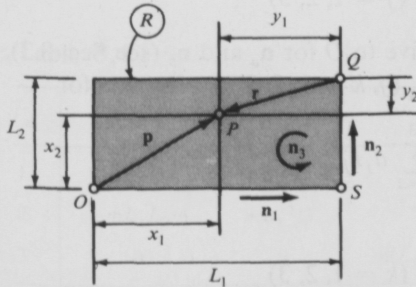


Figure 3.3.2

Similarly,  $\mathbf{I}_3^Q$ , the inertia vector of  $B$  with respect to point  $Q$  for  $\mathbf{n}_3$ , can be written (see Fig. 3.3.2 for  $\mathbf{r}$ )

$$\begin{aligned} \mathbf{I}_3^Q &= \int_R \rho \mathbf{r} \times (\mathbf{n}_3 \times \mathbf{r}) d\tau \\ &= \int_0^{L_2} \int_0^{L_1} \frac{m}{L_1 L_2} (-y_1 \mathbf{n}_1 - y_2 \mathbf{n}_2) \times [\mathbf{n}_3 \times (-y_1 \mathbf{n}_1 - y_2 \mathbf{n}_2)] dy_1 dy_2 \\ &= \frac{m}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} (y_1^2 + y_2^2) dy_1 dy_2 \mathbf{n}_3 = \frac{m}{3} (L_1^2 + L_2^2) \mathbf{n}_3 \end{aligned} \quad (14)$$

With the aid of these results, the inertia scalars  $I_{1j}^O$  and  $I_{3j}^Q$  ( $j = 1, 2, 3$ ) can be formed as

$$I_{11}^O = \mathbf{I}_1^O \cdot \mathbf{n}_1 = \frac{m L_2^2}{3} \quad I_{12}^O = -\frac{m L_1 L_2}{4} \quad I_{13}^O = 0 \quad (15)$$

$$I_{31}^Q = \mathbf{I}_3^Q \cdot \mathbf{n}_1 = 0 \quad I_{32}^Q = 0 \quad I_{33}^Q = \frac{m}{3} (L_1^2 + L_2^2) \quad (16)$$

Two of these inertia scalars are moments of inertia, namely,  $I_{11}^O$  and  $I_{33}^Q$ . The first is the moment of inertia of  $B$  about line  $OS$ ; the second is the moment of inertia of  $B$  about the line passing through  $Q$  and parallel to  $\mathbf{n}_3$ .

### 3.4 MUTUALLY PERPENDICULAR UNIT VECTORS

Knowledge of the inertia vectors  $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$  of a body  $B$  relative to a point  $O$  (see Sec. 3.3) for three mutually perpendicular unit vectors  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  enables one to find  $\mathbf{I}_a$ , the inertia vector of  $B$  relative to  $O$  for any unit vector  $\mathbf{n}_a$ , for

$$\mathbf{I}_a = \sum_{j=1}^3 a_j \mathbf{I}_j \quad (1)$$

where  $a_1, a_2, a_3$  are defined as

$$a_j \triangleq \mathbf{n}_a \cdot \mathbf{n}_j \quad (j = 1, 2, 3) \quad (2)$$

Similarly,  $I_{ab}$ , the inertia scalar of  $B$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$  (see Sec. 3.3), can be found easily when the inertia scalars  $I_{jk}$  ( $j, k = 1, 2, 3$ ) are known, for

$$I_{ab} = \sum_{j=1}^3 \sum_{k=1}^3 a_j I_{jk} b_k \quad (3)$$

where

$$b_k \triangleq \mathbf{n}_b \cdot \mathbf{n}_k \quad (k = 1, 2, 3) \quad (4)$$



**Derivations** It follows from Eq. (2) that (see Sec. 1.3)

$$\mathbf{n}_a = \sum_{j=1}^3 a_j \mathbf{n}_j \quad (5)$$

Consequently,

$$\begin{aligned} \mathbf{I}_a &= \sum_{i=1}^v m_i \mathbf{p}_i \times \left( \sum_{j=1}^3 a_j \mathbf{n}_j \times \mathbf{p}_i \right) \\ &= \sum_{j=1}^3 a_j \sum_{i=1}^v m_i \mathbf{p}_i \times (\mathbf{n}_j \times \mathbf{p}_i) \\ &= \sum_{j=1}^3 a_j \mathbf{I}_j \end{aligned} \quad (6)$$

which establishes the validity of Eq. (1). As for Eq. (3), note that Eq. (4) implies that (see Sec. 1.3)

$$\mathbf{n}_b = \sum_{k=1}^3 b_k \mathbf{n}_k \quad (7)$$

Hence,

$$\begin{aligned} I_{ab} &= \left( \sum_{j=1}^3 a_j \mathbf{I}_j \right) \cdot \sum_{k=1}^3 b_k \mathbf{n}_k \\ &= \sum_{j=1}^3 \sum_{k=1}^3 a_j \mathbf{I}_j \cdot \mathbf{n}_k b_k = \sum_{j=1}^3 \sum_{k=1}^3 a_j I_{jk} b_k \end{aligned} \quad (8)$$

**Example** Table 3.4.1 shows the inertia scalars  $I_{jk}$  of  $B$  relative to  $O$  for  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ), where  $B$  is the rectangular plate considered in the example in Sec. 3.3. To find the moment of inertia of  $B$  with respect to line  $OQ$  in Fig. 3.3.2, let  $\mathbf{n}_a$  be a unit vector parallel to this line, so that

$$\mathbf{n}_a = \frac{L_1 \mathbf{n}_1 + L_2 \mathbf{n}_2}{(L_1^2 + L_2^2)^{1/2}} \quad (9)$$

**Table 3.4.1**

$I_{jk}$	1	2	3
1	$mL_2^2/3$	$-mL_1L_2/4$	0
2	$-mL_1L_2/4$	$mL_1^2/3$	0
3	0	0	$m(L_1^2 + L_2^2)/3$

which means that, in accordance with Eq. (2),

$$a_1 = \frac{L_1}{(L_1^2 + L_2^2)^{1/2}} \quad a_2 = \frac{L_2}{(L_1^2 + L_2^2)^{1/2}} \quad a_3 = 0 \quad (10)$$

With  $b_k = a_k$  ( $k = 1, 2, 3$ ), Eq. (3) then yields [see also Eq. (3.3.4)]

$$\begin{aligned} I_a &= a_1^2 I_{11} + a_2^2 I_{22} + a_3^2 I_{33} + 2(a_1 a_2 I_{12} + a_2 a_3 I_{23} + a_3 a_1 I_{31}) \\ &= \frac{L_1^2 I_{11} + L_2^2 I_{22} + 2L_1 L_2 I_{12}}{(10) \quad L_1^2 + L_2^2} \end{aligned} \quad (11)$$

and use of Table 3.4.1 leads to

$$I_a = \frac{m L_1^2 L_2^2 / 3 + m L_1^2 L_2^2 / 3 - m L_1^2 L_2^2 / 2}{(11) \quad L_1^2 + L_2^2} = \frac{m(L_1 L_2)^2}{6(L_1^2 + L_2^2)} \quad (12)$$

### 3.5 INERTIA MATRIX, INERTIA DYADIC

The inertia scalars  $I_{jk}$  of a set  $S$  of particles relative to a point  $O$  for unit vectors  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ) can be used to define a square matrix  $I$ , called the *inertia matrix* of  $S$  relative to  $O$  for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , as follows:

$$I \triangleq \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad (1)$$

Suppose that  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are mutually perpendicular and that row matrices  $a$  and  $b$  are defined as

$$a \triangleq [a_1 \quad a_2 \quad a_3] \quad b \triangleq [b_1 \quad b_2 \quad b_3] \quad (2)$$

where  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$  are given by Eqs. (3.4.2) and (3.4.4), respectively. Then it follows immediately from Eq. (3.4.3) and the rules for multiplication of matrices that  $I_{ab}$ , the inertia scalar of  $S$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$ , is given by

$$I_{ab} = a I b^T \quad (3)$$

where  $b^T$  is the transpose of  $b$ , that is, the column matrix having  $b_i$  as the element in the  $i$ th row ( $i = 1, 2, 3$ ). Equation (3) is useful when inertia scalars are evaluated by means of machine computations and matrix multiplication routines are readily available.

The set  $S$  does not possess a unique inertia matrix relative to  $O$ , for, if  $\mathbf{n}_1', \mathbf{n}_2', \mathbf{n}_3'$  are mutually perpendicular unit vectors other than  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , and  $I'$  is defined as

$$I' \triangleq \begin{bmatrix} I_{11}' & I_{12}' & I_{13}' \\ I_{21}' & I_{22}' & I_{23}' \\ I_{31}' & I_{32}' & I_{33}' \end{bmatrix} \quad (4)$$

where  $I_{jk}'$  ( $j, k = 1, 2, 3$ ) are the inertia scalars of  $S$  relative to  $O$  for  $\mathbf{n}_j'$  and  $\mathbf{n}_k'$ , then  $I'$ , like  $I$ , is an inertia matrix of  $S$  relative to  $O$ , but  $I$  and  $I'$  are by no means equal to each other. Hence, when working with inertia matrices, one must keep in mind that each such matrix is associated with a specific vector basis. By way of contrast, the use of *dyadics* enables one to deal with certain topics involving inertia vectors and/or inertia scalars in a *basis-independent* way. To acquaint the reader with dyadics, we begin by focusing attention on two vectors,  $\mathbf{u}$  and  $\mathbf{v}$ , given by

$$\mathbf{u} = \mathbf{w} \cdot \mathbf{ab} + \mathbf{w} \cdot \mathbf{cd} + \dots \quad (5)$$

and

$$\mathbf{v} = \mathbf{ab} \cdot \mathbf{w} + \mathbf{cd} \cdot \mathbf{w} + \dots \quad (6)$$

respectively, where  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$ ,  $\dots$ , and  $\mathbf{w}$  are any vectors whatsoever. Equations (5) and (6) can be rewritten as

$$\mathbf{u} = \mathbf{w} \cdot (\mathbf{ab} + \mathbf{cd} + \dots) \quad (7)$$

and

$$\mathbf{v} = (\mathbf{ab} + \mathbf{cd} + \dots) \cdot \mathbf{w} \quad (8)$$

if it is understood that the right-hand members of Eqs. (7) and (8) have the same meanings as those of Eqs. (5) and (6), respectively. Furthermore, if the quantity within parentheses in Eqs. (7) and (8) is denoted by  $\mathbf{Q}$ , that is, if  $\mathbf{Q}$  is defined as

$$\mathbf{Q} \triangleq \mathbf{ab} + \mathbf{cd} + \dots \quad (9)$$

then Eqs. (7) and (8) give way to

$$\mathbf{u} = \mathbf{w} \cdot \mathbf{Q} \quad (10)$$

and

$$\mathbf{v} = \mathbf{Q} \cdot \mathbf{w} \quad (11)$$

respectively;  $\mathbf{Q}$  is called a *dyadic*; and Eqs. (5), (9), and (10) constitute a definition of *scalar premultiplication* of a dyadic with a vector, while Eqs. (6), (9), and (11) define the operation of *scalar postmultiplication* of a dyadic with a vector. In summary, then, a dyadic is a juxtaposition of vectors as in the right-hand member of Eq. (9), and scalar multiplication (pre- or post-) of a dyadic with a vector produces a vector.

A rather special dyadic  $\mathbf{U}$ , called the *unit dyadic*, comes to light in connection with Eq. (1.3.3), which suggests the definition

$$\mathbf{U} \triangleq \mathbf{a}_1 \mathbf{a}_1 + \mathbf{a}_2 \mathbf{a}_2 + \mathbf{a}_3 \mathbf{a}_3 \quad (12)$$

where  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  are mutually perpendicular unit vectors. In accordance with Eqs. (5), (9), and (10),

$$\mathbf{v} \cdot \mathbf{U} = \mathbf{v} \cdot \mathbf{a}_1 \mathbf{a}_1 + \mathbf{v} \cdot \mathbf{a}_2 \mathbf{a}_2 + \mathbf{v} \cdot \mathbf{a}_3 \mathbf{a}_3 = \mathbf{v} \quad (13)$$

(1.3.3)

Moreover, it follows from Eqs. (6), (9), and (11) that

$$\mathbf{U} \cdot \mathbf{v} = \underset{(12)}{\mathbf{a}_1 \mathbf{a}_1 \cdot \mathbf{v} + \mathbf{a}_2 \mathbf{a}_2 \cdot \mathbf{v} + \mathbf{a}_3 \mathbf{a}_3 \cdot \mathbf{v}} = \underset{(1.3.3)}{\mathbf{v}} \quad (14)$$

In other words,  $\mathbf{U}$  is a dyadic whose scalar product (pre- or post-) with any vector is equal to the vector itself.

Returning to the subject of inertia vectors and inertia scalars, let us consider the inertia vector  $\mathbf{I}_a$ , defined as in Eq. (3.3.1). This can be expressed as

$$\begin{aligned} \mathbf{I}_a &= \sum_{(3.3.1) \ i=1}^v m_i (\mathbf{n}_a \mathbf{p}_i^2 - \mathbf{n}_a \cdot \mathbf{p}_i \mathbf{p}_i) \\ &= \sum_{i=1}^v m_i (\mathbf{n}_a \cdot \underset{(13)}{\mathbf{U} \mathbf{p}_i^2} - \mathbf{n}_a \cdot \mathbf{p}_i \mathbf{p}_i) \end{aligned} \quad (15)$$

Hence, if a dyadic  $\mathbf{I}$  is defined as

$$\mathbf{I} \triangleq \sum_{i=1}^v m_i (\mathbf{U} \mathbf{p}_i^2 - \mathbf{p}_i \mathbf{p}_i) \quad (16)$$

then  $\mathbf{I}_a$  can be expressed as

$$\mathbf{I}_a = \mathbf{n}_a \cdot \mathbf{I} \quad (17)$$

The dyadic  $\mathbf{I}$  is called the *inertia dyadic* of  $S$  relative to  $O$ . When a body  $B$  is modeled as matter occupying a figure  $F$  (a curve, surface, or solid), then Eq. (16) is replaced with

$$\mathbf{I} \triangleq \int_F \rho (\mathbf{U} \mathbf{p}^2 - \mathbf{p} \mathbf{p}) d\tau \quad (18)$$

where  $\rho$ ,  $\mathbf{p}$ , and  $d\tau$  have the same meanings as in connection with Eq. (3.3.8).

Dot-multiplication of Eq. (17) with a unit vector  $\mathbf{n}_b$  leads to

$$\mathbf{I}_a \cdot \mathbf{n}_b = (\mathbf{n}_a \cdot \mathbf{I}) \cdot \mathbf{n}_b \quad (19)$$

In view of Eq. (3.3.2) and the fact that  $(\mathbf{n}_a \cdot \mathbf{I}) \cdot \mathbf{n}_b = \mathbf{n}_a \cdot (\mathbf{I} \cdot \mathbf{n}_b)$ , so that the parentheses in Eq. (19) are unnecessary, one thus finds that the inertia scalar  $I_{ab}$  can be expressed as

$$I_{ab} = \mathbf{n}_a \cdot \mathbf{I} \cdot \mathbf{n}_b \quad (20)$$

The inertia dyadic  $\mathbf{I}$  is said to be basis-independent because its definition, Eq. (16), does not involve any basis vectors. However, it can be expressed in various basis-dependent forms. For example, if  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are mutually perpendicular unit vectors,  $\mathbf{I}_j$  is the inertia vector of  $S$  relative to  $O$  for  $\mathbf{n}_j$  ( $j = 1, 2, 3$ ), and  $I_{jk}$  is the inertia scalar of  $S$  relative to  $O$  for  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ), then  $\mathbf{I}$  is given both by

$$\mathbf{I} = \sum_{j=1}^3 \mathbf{I}_j \mathbf{n}_j \quad (21)$$

and by

$$\mathbf{I} = \sum_{j=1}^3 \sum_{k=1}^3 I_{jk} \mathbf{n}_j \mathbf{n}_k \quad (22)$$

**Derivation** It may be verified that a relationship analogous to Eq. (1.3.3) applies to any dyadic  $\mathbf{Q}$ ; that is,

$$\mathbf{Q} = \mathbf{n}_1 \cdot \mathbf{Q} \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{Q} \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{Q} \mathbf{n}_3 \quad (23)$$

Consequently,

$$\begin{aligned} \mathbf{I} &= \mathbf{n}_1 \cdot \mathbf{I} \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{I} \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{I} \mathbf{n}_3 \\ &= \mathbf{I}_1 \mathbf{n}_1 + \mathbf{I}_2 \mathbf{n}_2 + \mathbf{I}_3 \mathbf{n}_3 \end{aligned} \quad (24)$$

(17)

in agreement with Eq. (21). As for Eq. (22), refer to Eq. (1.3.3) to write

$$\begin{aligned} \mathbf{I}_j &= \mathbf{I}_j \cdot \mathbf{n}_1 \mathbf{n}_1 + \mathbf{I}_j \cdot \mathbf{n}_2 \mathbf{n}_2 + \mathbf{I}_j \cdot \mathbf{n}_3 \mathbf{n}_3 \\ &= I_{j1} \mathbf{n}_1 + I_{j2} \mathbf{n}_2 + I_{j3} \mathbf{n}_3 \quad (j = 1, 2, 3) \end{aligned} \quad (25)$$

(3.3.2)

and substitute these expressions for  $\mathbf{I}_1$ ,  $\mathbf{I}_2$ , and  $\mathbf{I}_3$  into Eq. (24) to obtain

$$\begin{aligned} \mathbf{I} &= (I_{11} \mathbf{n}_1 + I_{12} \mathbf{n}_2 + I_{13} \mathbf{n}_3) \mathbf{n}_1 \\ &\quad + (I_{21} \mathbf{n}_1 + I_{22} \mathbf{n}_2 + I_{23} \mathbf{n}_3) \mathbf{n}_2 \\ &\quad + (I_{31} \mathbf{n}_1 + I_{32} \mathbf{n}_2 + I_{33} \mathbf{n}_3) \mathbf{n}_3 \end{aligned} \quad (26)$$

which is seen to agree with Eq. (22).

**Example** If  $S$  is a set of particles  $P_1, \dots, P_v$  of masses  $m_1, \dots, m_v$ , respectively, moving in a reference frame  $A$  with velocities  ${}^A\mathbf{v}^{P_1}, \dots, {}^A\mathbf{v}^{P_v}$  (see Sec. 2.6), then a vector  ${}^A\mathbf{H}^{S/O}$ , called the *angular momentum* of  $S$  relative to  $O$  in  $A$ , is defined as

$${}^A\mathbf{H}^{S/O} \triangleq \sum_{i=1}^v m_i \mathbf{p}_i \times {}^A\mathbf{v}^{P_i} \quad (27)$$

where  $\mathbf{p}_i$  is the position vector from a point  $O$  to  $P_i$  ( $i = 1, \dots, v$ ). The point  $O$  need not be fixed in  $A$ ; for example, it can be the mass center  $S^*$  of  $S$ , in which case  ${}^A\mathbf{H}^{S/O}$  becomes  ${}^A\mathbf{H}^{S/S^*}$  and is called the *central angular momentum* of  $S$  in  $A$ .

If the particles of  $S$  form a rigid body  $B$ , then  $\mathbf{H}$ , the central angular momentum of  $B$  in  $A$ , can be expressed as

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} \quad (28)$$

where  $\mathbf{I}$ , called the *central inertia dyadic* of  $B$ , is the inertia dyadic of  $B$  relative to the mass center  $B^*$  of  $B$ , and  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $A$ . To verify

that  $\mathbf{H}$  can be written as in Eq. (28), let  $\mathbf{r}_i$  be the position vector from  $B^*$  to  $P_i$  and refer to Eqs. (27) and (2.7.1) to write

$$\begin{aligned}\mathbf{H} &= \sum_{i=1}^v m_i \mathbf{r}_i \times (\mathbf{A}_{\mathbf{v}B^*} + \boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \left( \sum_{i=1}^v m_i \mathbf{r}_i \right) \times \mathbf{A}_{\mathbf{v}B^*} + \sum_{i=1}^v m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\end{aligned}\quad (29)$$

Then note that

$$\sum_{i=1}^v m_i \mathbf{r}_i \stackrel{(3.1.1)}{=} 0 \quad (30)$$

while

$$\begin{aligned}\sum_{i=1}^v m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) &= \sum_{i=1}^v m_i (\mathbf{r}_i^2 \boldsymbol{\omega} - \mathbf{r}_i \mathbf{r}_i \cdot \boldsymbol{\omega}) \\ &= \sum_{i=1}^v m_i (\mathbf{r}_i^2 \mathbf{U} - \mathbf{r}_i \mathbf{r}_i) \cdot \boldsymbol{\omega} = \mathbf{I} \cdot \boldsymbol{\omega} \quad (16)\end{aligned}\quad (31)$$

If  $\omega_1, \omega_2, \omega_3$  are defined as

$$\omega_i \triangleq \boldsymbol{\omega} \cdot \mathbf{n}_i \quad (i = 1, 2, 3) \quad (32)$$

then it follows directly from Eqs. (28), (22), and (32) that

$$\begin{aligned}\mathbf{H} &= \sum_{j=1}^3 \sum_{k=1}^3 I_{jk} \mathbf{n}_j \mathbf{n}_k \cdot \sum_{i=1}^3 \omega_i \mathbf{n}_i \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 I_{jk} \omega_i \mathbf{n}_j \mathbf{n}_k \cdot \mathbf{n}_i\end{aligned}\quad (33)$$

so that, since  $\mathbf{n}_k \cdot \mathbf{n}_i$  vanishes except when  $i = k$ , in which case it is equal to unity,  $\mathbf{H}$  can be written

$$\mathbf{H} \stackrel{(33)}{=} \sum_{j=1}^3 \sum_{k=1}^3 I_{jk} \omega_k \mathbf{n}_j \quad (34)$$

This is a useful relationship. It reduces to the convenient form given in Eq. (2.3.5) if  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are chosen in such a way that  $I_{12}, I_{23}$ , and  $I_{31}$  vanish. The fact that it is always possible to choose  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  in this way is established in Sec. 3.8.

### 3.6 PARALLEL AXES THEOREMS

The inertia dyadic  $\mathbf{I}^{S/O}$  of a set  $S$  of  $v$  particles  $P_1, \dots, P_v$  relative to a point  $O$  (see Sec. 3.5) is related in a simple way to the *central inertia dyadic*  $\mathbf{I}^{S/S^*}$  of  $S$ , that is, the inertia dyadic of  $S$  relative to the mass center  $S^*$  of  $S$ . Specifically,

$$\mathbf{I}^{S/O} = \mathbf{I}^{S/S^*} + \mathbf{I}^{S^*/O} \quad (1)$$

where  $\mathbf{I}^{S^*/O}$  denotes the inertia dyadic relative to  $O$  of a (fictitious) particle situated at  $S^*$  and having a mass equal to the total mass of  $S$ . Similarly, if  $I^{S/O}$  and  $I^{S/S^*}$

are inertia matrices of  $S$  relative to  $O$  and  $S^*$ , respectively, for unit vectors  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , and  $I^{S^*/O}$  is the inertia matrix relative to  $O$  of a particle having the same mass as  $S$  and situated at  $S^*$ , also for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , then

$$I^{S/O} = I^{S/S^*} + I^{S^*/O} \quad (2)$$

Moreover, analogous relationships apply to inertia vectors, products of inertia, and moments of inertia (see Sec. 3.3); that is,

$$\mathbf{I}_a^{S/O} = \mathbf{I}_a^{S/S^*} + \mathbf{I}_a^{S^*/O} \quad (3)$$

$$I_{ab}^{S/O} = I_{ab}^{S/S^*} + I_{ab}^{S^*/O} \quad (4)$$

and

$$I_a^{S/O} = I_a^{S/S^*} + I_a^{S^*/O} \quad (5)$$

The quantities  $I_{ab}^{S/S^*}$  and  $I_a^{S/S^*}$  are called *central inertia scalars*. Equations (3)–(5) are referred to as parallel axes theorems because one can associate certain parallel lines with each of these equations, such as, in the case of Eq. (5), the two lines that are parallel to  $\mathbf{n}_a$  and pass through  $O$  and  $S^*$ .

**Derivations** Let  $\mathbf{p}_i$  and  $\mathbf{r}_i$  be the position vectors from  $O$  to  $P_i$  and from  $S^*$  to  $P_i$  ( $i = 1, \dots, v$ ), respectively; note that Eq. (3.1.1) is satisfied if  $m_i$  denotes the mass of  $P_i$ , and that

$$\mathbf{p}_i = \mathbf{p}^* + \mathbf{r}_i \quad (6)$$

where  $\mathbf{p}^*$  is the position vector from  $O$  to  $S^*$ . Then  $I^{S/O}$ ,  $I^{S/S^*}$ , and  $I^{S^*/O}$  are given by

$$I^{S/O} = \sum_{(3.5.16) \ i=1}^v m_i (\mathbf{U} \mathbf{p}_i^2 - \mathbf{p}_i \mathbf{p}_i) \quad (7)$$

$$I^{S/S^*} = \sum_{(3.5.16) \ i=1}^v m_i (\mathbf{U} \mathbf{r}_i^2 - \mathbf{r}_i \mathbf{r}_i) \quad (8)$$

and

$$I^{S^*/O} = \left( \sum_{(3.5.16) \ i=1}^v m_i \right) (\mathbf{U} \mathbf{p}^{*2} - \mathbf{p}^* \mathbf{p}^*) \quad (9)$$

Hence,

$$\begin{aligned} I^{S/O} &= \sum_{(7) \ i=1}^v m_i [\mathbf{U} (\mathbf{p}^* + \mathbf{r}_i)^2 - (\mathbf{p}^* + \mathbf{r}_i)(\mathbf{p}^* + \mathbf{r}_i)] \\ &= \sum_{i=1}^v m_i [\mathbf{U} (\mathbf{p}^{*2} + 2\mathbf{p}^* \cdot \mathbf{r}_i + \mathbf{r}_i^2) - \mathbf{p}^* \mathbf{p}^* \\ &\quad - \mathbf{r}_i \mathbf{p}^* - \mathbf{p}^* \mathbf{r}_i - \mathbf{r}_i \mathbf{r}_i] \\ &= \sum_{i=1}^v m_i (\mathbf{U} \mathbf{r}_i^2 - \mathbf{r}_i \mathbf{r}_i) + \left( \sum_{i=1}^v m_i \right) (\mathbf{U} \mathbf{p}^{*2} - \mathbf{p}^* \mathbf{p}^*) \\ &\quad + 2\mathbf{U} \mathbf{p}^* \cdot \left( \sum_{i=1}^v m_i \mathbf{r}_i \right) - \left( \sum_{i=1}^v m_i \mathbf{r}_i \right) \mathbf{p}^* - \mathbf{p}^* \left( \sum_{i=1}^v m_i \mathbf{r}_i \right) \end{aligned} \quad (10)$$

and use of Eqs. (8), (9), and (3.1.1) leads directly to Eq. (1).

Premultiplication of Eq. (1) with a unit vector  $\mathbf{n}_a$  yields Eq. (3) when Eq. (3.5.17) is taken into account; similarly, use of Eq. (3.3.2) after dot-multiplication of Eq. (3) with  $\mathbf{n}_b$  and  $\mathbf{n}_a$  leads to Eqs. (4) and (5), respectively. Finally, Eq. (2) is an immediate consequence of the fact that the elements of  $I^{S/O}$ ,  $I^{S/S^*}$ , and  $I^{S^*/O}$  satisfy Eqs. (4) and (5).

**Example** In Fig. 3.6.1,  $C$  represents a wing of a delta-wing aircraft, modeled as a uniform, thin, right-triangular, plate of mass  $m$ . The central inertia matrix of  $C$  for the unit vectors  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ ,  $\mathbf{n}_3$  is given by

$$I^{C/C^*} = mc^2 \begin{bmatrix} \frac{9}{2} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad (11)$$

The product of inertia of  $C$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$  is to be determined (see Fig. 3.6.1 for point  $O$  and the unit vectors  $\mathbf{n}_a$  and  $\mathbf{n}_b$ ).

The unit vectors  $\mathbf{n}_a$  and  $\mathbf{n}_b$  can be expressed, respectively, as

$$\mathbf{n}_a = \frac{-\mathbf{n}_1 + 3\mathbf{n}_2}{\sqrt{10}} \quad \mathbf{n}_b = -\mathbf{n}_1 \quad (12)$$

Hence, if matrices  $a$  and  $b$  are defined as

$$a \triangleq [-1/\sqrt{10} \quad 3/\sqrt{10} \quad 0] \quad b \triangleq [-1 \quad 0 \quad 0] \quad (13)$$

then the central product of inertia of  $C$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$  is given by

$$I_{ab}^{C/C^*} = \underset{(3.5.3)}{a} I_{C/C^*} \underset{(11,13)}{b}^T = \frac{9mc^2}{4\sqrt{10}} \quad (14)$$

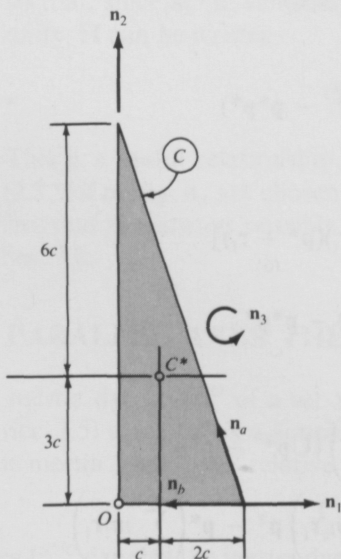


Figure 3.6.1



The position vector  $\mathbf{p}$  from  $O$  to  $C^*$  is

$$\mathbf{p}^* = c\mathbf{n}_1 + 3c\mathbf{n}_2 \quad (15)$$

Consequently, the product of inertia of a particle of mass  $m$  situated at  $C^*$ , relative to  $O$ , for  $\mathbf{n}_a$  and  $\mathbf{n}_b$ , is

$$I_{ab}^{C^*/O} \underset{(3.3.3)}{=} m(\mathbf{p}^* \times \mathbf{n}_a) \cdot (\mathbf{p}^* \times \mathbf{n}_b) \underset{(12,15)}{=} \frac{18mc^2}{\sqrt{10}} \quad (16)$$

and

$$I_{ab}^{C/O} \underset{(4)}{=} I_{ab}^{C/C^*} + I_{ab}^{C^*/O} \underset{(14,16)}{=} \frac{81mc^2}{4\sqrt{10}} \quad (17)$$

### 3.7 EVALUATION OF INERTIA SCALARS

Generally speaking, the most direct way to find inertia scalars (see Sec. 3.3) of a set of particles (including sets consisting of a single particle) is to use Eq. (3.3.3). Exceptions to this rule arise when the value of an appropriate central inertia scalar (see Sec. 3.6) is known, in which event one can appeal to Eq. (3.6.4), or when a suitable inertia vector (see Sec. 3.3), inertia matrix, or inertia dyadic (see Sec. 3.5) is available, making it possible to use Eqs. (3.3.2), (3.5.3), or (3.5.20). In a sense, the opposite applies to the evaluation of inertia scalars of a body modeled as matter distributed along a curve, over a surface, or throughout a solid. Here, one should use Eq. (3.3.9), which is analogous to Eq. (3.3.3), only when one cannot find any tabulated information regarding the inertia properties of the body under consideration, which is likely to be the case only when the body has a variable mass density or occupies an irregular figure. When, as happens more often in engineering practice, the mass density is uniform and the figure in question is one of those considered in an available table of inertia properties, the best way to proceed is to use, first, Eq. (3.4.3), (3.5.3), or (3.5.20) with  $I_{jk}$  ( $j, k = 1, 2, 3$ ) representing central inertia scalars, and then to turn to Eq. (3.6.4), evaluating the last term in this equation with the aid of Eq. (3.3.3) after setting  $v = 1$  and  $m_1$  equal to the mass of the body. Appendix I contains information making it possible to proceed in this way in connection with a number of figures chosen from among those encountered most frequently in engineering practice.

When an object can be regarded as composed of bodies  $B_1, \dots, B_v$ , and an inertia scalar of this object with respect to a point  $P$  is to be found, one can use the above procedure to find the corresponding inertia scalar of each of  $B_1, \dots, B_v$  with respect to  $P$ , and then simply add these inertia scalars. This follows from Eq. (3.3.3) and the associativity of scalar addition.

Differentials sometimes can be used to advantage in connection with inertia calculations involving objects that can be regarded as thin-walled counterparts of bodies having known inertia properties. Consider, for example, a closed cubical container  $C$  having sides of length  $L$ , and let  $k$  be the radius of gyration of  $C$  with

respect to a line passing through  $C^*$ , the mass center of  $C$ , and parallel to an edge of  $C$ . Suppose, further, that the walls of  $C$  are thin. To determine  $k$ , let  $\rho$  be the mass density of a uniform solid cube  $S$  having sides of length  $L$ . Then  $m$ , the mass of  $S$ , and  $I$ , the moment of inertia of  $S$  about a line parallel to an edge of  $S$  and passing through  $S^*$ , the mass center of  $S$ , are given by (see Appendix I)

$$m = \rho L^3 \quad (1)$$

and

$$I = \frac{\rho L^5}{6} \quad (2)$$

respectively, and the differentials of  $m$  and  $I$  are

$$dm = 3\rho L^2 dL \quad (3)$$

(1)

and

$$dI = \frac{5\rho L^4}{6} dL \quad (4)$$

(2)

Consequently, attributing to  $C$  the wall thickness  $dL/2$ , and hence the mass  $dm$  and the moment of inertia  $dI$ , one can write

$$\frac{5\rho L^4}{6} dL = 3\rho L^2 dL k^2 \quad (5)$$

(3.3.6) (3) (4)

from which it follows immediately that

$$k = \frac{1}{3} \sqrt{\frac{5}{2}} L \quad (6)$$

The same result is obtained with somewhat more effort, but quite revealingly, by reasoning as follows

When  $C$  is regarded as formed by removing from a solid cube  $S_1$  of mass density  $\rho$  and having sides of length  $L_1$  a solid cube  $S_2$  of the same mass density and having sides of length  $L_2$ , the mass  $m$  and moment of inertia  $I$  of  $C$  about a line passing through  $C^*$  and parallel to an edge of  $C$  can be expressed as

$$m = \rho L_1^3 - \rho L_2^3 = \rho(L_1^3 - L_2^3) \quad (7)$$

and

$$I = \rho \frac{L_1^5}{6} - \rho \frac{L_2^5}{6} = \frac{\rho}{6} (L_1^5 - L_2^5) \quad (8)$$

Consequently,  $\bar{k}$ , the radius of gyration of interest, is given by

$$\bar{k} = \left( \frac{I}{m} \right)^{1/2} = \left[ \frac{L_1^5 - L_2^5}{6(L_1^3 - L_2^3)} \right]^{1/2} \quad (9)$$

(3.3.6) (7,8)

Furthermore, in order for  $C$  to have sides of length  $L$  and walls of thickness  $t/2$ ,  $L_1$  and  $L_2$  must be given by

$$L_1 = L \quad L_2 = L - t \quad (10)$$

which leads to

$$\begin{aligned} \bar{k}_{(9,10)} &= \frac{1}{\sqrt{6}} \left[ \frac{L^5 - (L-t)^5}{L^3 - (L-t)^3} \right]^{1/2} \\ &= \frac{1}{\sqrt{6}} \left( \frac{5L^4 - 10L^3t + 10L^2t^2 - 5Lt^3 + t^4}{3L^2 - 3Lt + t^2} \right)^{1/2} \end{aligned} \quad (11)$$

and

$$\lim_{t \rightarrow 0} \bar{k}_{(11)} = \frac{1}{3} \sqrt{\frac{5}{2}} L \quad (12)$$

**Example** The radius of gyration  $k$  of the body  $B$  described in the example in Sec. 3.2 and shown in Fig. 3.2.1 is to be found with respect to a line that is parallel to  $X_2$  and passes through  $B^*$ , the mass center of  $B$ .

Regarding  $B$  as composed of the three bodies  $B_1$ ,  $B_2$ ,  $B_3$  introduced previously, and noting that the masses of  $B_1$ ,  $B_2$ ,  $B_3$  are [see Eqs. (3.2.3)]

$$m_1 = m \quad m_2 = \left(\frac{\pi}{2}\right)m \quad m_3 = 10\left(1 + \frac{\pi}{2}\right)m \quad (13)$$

while  $\mathbf{p}^*$ , the position vector from  $E$  to  $B^*$ , is given by

$$\mathbf{p}^*_{(3.2.4)} = (1.82\mathbf{a}_1 + 2.03\mathbf{a}_2 + 0.318\mathbf{a}_3)L \quad (14)$$

and the mass centers of  $B_1$ ,  $B_2$ ,  $B_3$  are the points  $B_1^*$ ,  $B_2^*$ ,  $B_3^*$  shown in Fig. 3.2.2, begin by forming expressions for the moment of inertia of  $B_i$  with respect to a line passing through  $B_i^*$  ( $i = 1, 2, 3$ ) and parallel to  $\mathbf{a}_2$ .

For  $B_1$ , from Eqs. (13), Fig. 3.2.2, and Appendix I,

$$I_{2^{B_1/B_1^*}} = \frac{m(6L)^2}{12}_{(13)} = 3mL^2 \quad (15)$$

To deal with  $B_2$ , introduce unit vectors  $\mathbf{n}_a$  and  $\mathbf{n}_b$  as shown in Fig. 3.2.2, and refer to Appendix I to write

$$I_{a^{B_2/B_2^*}} = \frac{(\pi/2)m(6L)^2}{2} \left[ 1 + \frac{1}{\pi/2} - \frac{1}{(\pi/4)^2} \right] = 0.438mL^2 \quad (16)$$

$$I_{b^{B_2/B_2^*}} = \frac{(\pi/2)m(6L)^2}{2} \left( 1 - \frac{1}{\pi/2} \right) = 10.3mL^2 \quad (17)$$

and then appeal to Eq. (3.4.3) to obtain

$$I_2^{B_2/B_2^*} = \left(\frac{1}{\sqrt{2}}\right)^2 I_a^{B_2/B_2^*} + \left(\frac{1}{\sqrt{2}}\right)^2 I_b^{B_2/B_2^*} = 5.36mL^2 \quad (18)$$

(16,17)

Finally, refer to Appendix I to verify that

$$I_2^{B_3/B_3^*} = 10\left(1 + \frac{\pi}{2}\right)m \frac{(6L)^2}{18} = 51.4mL^2 \quad (19)$$

Next, determine  $I_2^{B_i^*/B^*}$  ( $i = 1, 2, 3$ ) by using Eq. (3.3.3) with  $v = 1$ , that is, by expressing  $I_2^{B_i^*/B^*}$  as

$$I_2^{B_i^*/B^*} = m_i[(\mathbf{p}^* - \mathbf{p}_i) \times \mathbf{a}_2]^2 \quad (i = 1, 2, 3) \quad (20)$$

where  $\mathbf{p}_i$  is the position vector from  $E$  (see Fig. 3.2.2) to  $B_i^*$  ( $i = 1, 2, 3$ ). Specifically,

$$\begin{aligned} I_2^{B_1^*/B^*} &= m[(\mathbf{p}^* - 3L\mathbf{a}_3) \times \mathbf{a}_2]^2 \\ (20) \quad &= mL^2(1.82\mathbf{a}_3 + 2.68\mathbf{a}_1)^2 = 10.5mL^2 \end{aligned} \quad (21)$$

(14)

$$I_2^{B_2^*/B^*} = \left(\frac{\pi}{2}\right)m \left\{ \left[ \mathbf{p}^* - \left(\frac{12L}{\pi}\right)(\mathbf{a}_2 + \mathbf{a}_3) \right] \times \mathbf{a}_2 \right\}^2 = 24.5mL^2 \quad (22)$$

(20) (14)

$$I_2^{B_3^*/B^*} = 10\left(1 + \frac{\pi}{2}\right)m \{[(\mathbf{p}^* - 2L(\mathbf{a}_1 + \mathbf{a}_2)] \times \mathbf{a}_2\}^2 = 3.43mL^2 \quad (23)$$

(20) (14)

Now refer to Eq. (3.6.5) to evaluate  $I_2^{B_i/B^*}$  ( $i = 1, 2, 3$ ) as

$$I_2^{B_i/B^*} = I_2^{B_i/B_i^*} + I_2^{B_i^*/B^*} \quad (i = 1, 2, 3) \quad (24)$$

which gives

$$I_2^{B_1/B^*} = 3mL^2 + 10.5mL^2 = 13.5mL^2 \quad (25)$$

(24) (15) (21)

$$I_2^{B_2/B^*} = 5.36mL^2 + 24.5mL^2 = 29.9mL^2 \quad (26)$$

(24) (18) (22)

$$I_2^{B_3/B^*} = 51.4mL^2 + 3.43mL^2 = 54.8mL^2 \quad (27)$$

(24) (19) (23)

Consequently,

$$I_2^{B/B^*} = I_2^{B_1/B^*} + I_2^{B_2/B^*} + I_2^{B_3/B^*} = 98.2mL^2 \quad (28)$$

(25-27)

and

$$k = \left( \frac{I_2^{B/B^*}}{m_1 + m_2 + m_3} \right)^{1/2} = 1.86L \quad (29)$$

(3.3.6) (13,28)

### 3.8 PRINCIPAL MOMENTS OF INERTIA

In general, the inertia vector  $\mathbf{I}_z$  (see Sec. 3.3) is not parallel to  $\mathbf{n}_z$ . When  $\mathbf{n}_z$  is a unit vector such that  $\mathbf{I}_z$  is parallel to  $\mathbf{n}_z$ , the line  $L_z$  passing through  $O$  and parallel to  $\mathbf{n}_z$  is called a *principal axis* of  $S$  for  $O$ , the plane  $P_z$  passing through  $O$  and normal to  $\mathbf{n}_z$  is called a *principal plane* of  $S$  for  $O$ , the moment of inertia  $I_z$  of  $S$  with respect to  $L_z$  is called a *principal moment of inertia* of  $S$  for  $O$ , and the radius of gyration of  $S$  with respect to  $L_z$  is called a *principal radius of gyration* of  $S$  for  $O$ . When the point  $O$  under consideration is the mass center of  $S$ , one speaks of *central* principal axes, central principal planes, central principal moments of inertia, and central principal radii of gyration.

When  $\mathbf{n}_z$  is parallel to a principal axis of  $S$  for  $O$ , the inertia vector  $\mathbf{I}_z$  of  $S$  relative to  $O$  for  $\mathbf{n}_z$  can be expressed as

$$\mathbf{I}_z = I_z \mathbf{n}_z \quad (1)$$

and, if  $\mathbf{n}_y$  is any unit vector perpendicular to  $\mathbf{n}_z$ , then the product of inertia of  $S$  relative to  $O$  for  $\mathbf{n}_y$  and  $\mathbf{n}_z$  vanishes; that is,

$$I_{yz} = 0 \quad (2)$$

Suppose that  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are mutually perpendicular unit vectors, each parallel to a principal axis of  $S$  for  $O$ , and  $\mathbf{n}_a$  and  $\mathbf{n}_b$  are any two unit vectors. Then, if  $a_i$  and  $b_i$  are defined as

$$a_i \triangleq \mathbf{n}_a \cdot \mathbf{n}_i \quad b_i \triangleq \mathbf{n}_b \cdot \mathbf{n}_i \quad (i = 1, 2, 3) \quad (3)$$

the inertia scalar  $I_{ab}$  of  $S$  relative to  $O$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$  is given by

$$I_{ab} = a_1 I_1 b_1 + a_2 I_2 b_2 + a_3 I_3 b_3 \quad (4)$$

where  $I_1, I_2, I_3$  are the principal moments of inertia of  $S$  for  $O$  associated with  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , respectively. This relationship is considerably simpler than its more general counterpart, Eq. (3.4.3), which applies even when  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are not parallel to principal axes of  $S$  for  $O$ . Similarly, Eqs. (3.5.1) and (3.5.22) reduce to

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad (5)$$

and

$$\mathbf{I} = I_1 \mathbf{n}_1 \mathbf{n}_1 + I_2 \mathbf{n}_2 \mathbf{n}_2 + I_3 \mathbf{n}_3 \mathbf{n}_3 \quad (6)$$

respectively, when  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are parallel to principal axes of  $S$  for  $O$ .

For every set of particles there exists at least one set of three mutually perpendicular principal axes for every point in space. To locate principal axes of a set  $S$  of particles for a point  $O$ , and to determine the associated principal moments of inertia, one exploits the following facts.

If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are any mutually perpendicular unit vectors,  $I_{jk}$  ( $j, k = 1, 2, 3$ ) are the associated inertia scalars of  $S$  for  $O$ , and  $I_z$  is a principal moment of inertia of  $S$  for  $O$ , then  $I_z$  satisfies the cubic *characteristic equation*

$$\begin{vmatrix} (I_1 - I_z) & I_{12} & I_{13} \\ I_{21} & (I_2 - I_z) & I_{23} \\ I_{31} & I_{32} & (I_3 - I_z) \end{vmatrix} = 0 \quad (7)$$

A unit vector  $\mathbf{n}_z$  is parallel to the principal axis associated with the principal moment of inertia  $I_z$  if

$$\mathbf{n}_z = z_1 \mathbf{n}_1 + z_2 \mathbf{n}_2 + z_3 \mathbf{n}_3 \quad (8)$$

where  $z_1, z_2, z_3$  satisfy the four equations

$$\sum_{j=1}^3 z_j I_{jk} = I_z z_k \quad (k = 1, 2, 3) \quad (9)$$

$$z_1^2 + z_2^2 + z_3^2 = 1 \quad (10)$$

When Eq. (7) has precisely two equal roots, every line passing through  $O$  and lying in the principal plane of  $S$  for  $O$  corresponding to the remaining root of Eq. (7) is a principal axis of  $S$  for  $O$ ; when Eq. (7) has three equal roots, every line passing through  $O$  is a principal axis of  $S$  for  $O$ .

Once a principal plane  $P_z$  of  $S$  for  $O$  has been identified, principal axes of  $S$  for  $O$  lying in  $P_z$ , as well as the associated principal moments of inertia of  $S$  for  $O$ , can be found without solving a cubic equation. If  $\mathbf{n}_a$  and  $\mathbf{n}_b$  are any two unit vectors parallel to  $P_z$  and perpendicular to each other, while  $I_a, I_b$ , and  $I_{ab}$  are the associated moments of inertia and product of inertia of  $S$  for  $O$ , then the angle  $\theta$  between  $\mathbf{n}_a$  and each of the principal axes in question satisfies the equation

$$\tan 2\theta = \frac{2I_{ab}}{I_a - I_b} \quad (I_a \neq I_b) \quad (11)$$

and the associated principal moments of inertia of  $S$  for  $O$ , say,  $I_x$  and  $I_y$ , are given by

$$I_x, I_y = \frac{I_a + I_b}{2} \pm \left[ \left( \frac{I_a - I_b}{2} \right)^2 + I_{ab}^2 \right]^{1/2} \quad (12)$$

If  $I_a = I_b$ , then every line passing through  $O$  and lying in  $P_z$  is a principal axis of  $S$  for  $O$ , and the associated principal moments of inertia of  $S$  for  $O$  all have the value  $I_a$ .

Frequently, principal planes can be located by means of symmetry considerations. For example, when all particles of  $S$  lie in a plane, then this plane is a principal plane of  $S$  for every point of the plane, for the inertia vector of  $S$  relative to any point of the plane is then normal to the plane.

The eigenvalues of the matrix  $I$  defined in Eq. (3.5.1) are principal moments of inertia of  $S$  for  $O$  because these eigenvalues satisfy Eq. (7); and, if a matrix  $[e_1 \ e_2 \ e_3]$

is an eigenvector of  $I$ , and  $e_1^2 + e_2^2 + e_3^2 = 1$ , then  $e_1 \mathbf{n}_1 + e_2 \mathbf{n}_2 + e_3 \mathbf{n}_3$  is a unit vector parallel to a principal axis of  $S$  for  $O$  because  $e_1, e_2, e_3$  then satisfy Eqs. (9) whenever  $I_z$  is an eigenvalue of  $I$ . Hence, when computer routines for finding eigenvalues and eigenvectors of a symmetric matrix are available, these can be used directly to determine principal moments of inertia and to locate the associated principal axes.

**Derivations** When  $\mathbf{n}_z$  is parallel to a principal axis of  $S$  for  $O$ , and hence  $\mathbf{I}_z$  is parallel to  $\mathbf{n}_z$ , then there exists a quantity  $\lambda$  such that

$$\lambda \mathbf{n}_z = \mathbf{I}_z \quad (13)$$

Dot-multiplication of this equation with  $\mathbf{n}_z$  gives

$$\lambda = \mathbf{I}_z \cdot \mathbf{n}_z \underset{(3.3.2)}{=} I_z \quad (14)$$

and substitution from this equation into Eq. (13) yields Eq. (1). Furthermore, if  $\mathbf{n}_y$  is any unit vector perpendicular to  $\mathbf{n}_z$ , then dot-multiplication of Eq. (1) with  $\mathbf{n}_y$  and use of Eqs. (3.3.2) and (3.3.4) leads to Eq. (2).

If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are mutually perpendicular unit vectors, each parallel to a principal axis of  $S$  for  $O$ , then, in accordance with Eq. (2),

$$I_{12} = I_{23} = I_{31} = 0 \quad (15)$$

and substitution from these equations into Eqs. (3.4.3), (3.5.1), and (3.5.22) establishes the validity of Eqs. (4), (5), and (6), respectively.

If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are any mutually perpendicular unit vectors,  $I_{jk}$  ( $j, k = 1, 2, 3$ ) are the associated inertia scalars of  $S$  for  $O$ ,  $\mathbf{n}_z$  is any unit vector, and  $z_1, z_2, z_3$  are defined as

$$z_i \triangleq \mathbf{n}_z \cdot \mathbf{n}_i \quad (i = 1, 2, 3) \quad (16)$$

then

$$\mathbf{n}_z \underset{(1.3.1)}{=} \sum_{k=1}^3 z_k \mathbf{n}_k \quad (17)$$

and

$$\mathbf{I}_z \underset{(3.4.1)}{=} \sum_{j=1}^3 z_j \mathbf{I}_j \quad (18)$$

Now

$$\mathbf{I}_j \underset{(1.3.3)}{=} \sum_{k=1}^3 \mathbf{I}_j \cdot \mathbf{n}_k \mathbf{n}_k \underset{(3.3.2)}{=} \sum_{k=1}^3 I_{jk} \mathbf{n}_k \quad (j = 1, 2, 3) \quad (19)$$

Hence,

$$\mathbf{I}_z \underset{(18,19)}{=} \sum_{j=1}^3 \sum_{k=1}^3 z_j I_{jk} \mathbf{n}_k \quad (20)$$

and  $\mathbf{n}_z$  is parallel to a principal axis of  $S$  for  $O$  if

$$\sum_{j=1}^3 \sum_{k=1}^3 z_j I_{jk} \mathbf{n}_k = I_z \sum_{k=1}^3 z_k \mathbf{n}_k \quad (21)$$

(20) (1) (17)

This vector equation is equivalent to the three scalar equations

$$\sum_{j=1}^3 z_j I_{jk} = I_z z_k \quad (k = 1, 2, 3) \quad (22)$$

Also,  $z_1, z_2, z_3$  satisfy the equation

$$z_1^2 + z_2^2 + z_3^2 = 1 \quad (23)$$

because  $\mathbf{n}_z$  is a unit vector. [Equations (22) and (23) are Eqs. (9) and (10), respectively.] As Eqs. (22) are linear and homogeneous in  $z_1, z_2, z_3$ , and as these three quantities cannot all vanish, for this would violate Eq. (23), Eqs. (22) can be satisfied only if the determinant of the coefficients of  $z_1, z_2, z_3$  vanishes, that is, if Eq. (7) is satisfied. Now, Eq. (7) is cubic in  $I_z$ . Hence, there exist three values of  $I_z$  (not necessarily distinct) that satisfy Eq. (7). It will now be shown that all such values of  $I_z$  are real.

Let  $A, B, \alpha_j$ , and  $\beta_j$  ( $j = 1, 2, 3$ ) be real quantities such that

$$I_z = A + iB \quad (24)$$

and

$$z_j = \alpha_j + i\beta_j \quad (j = 1, 2, 3) \quad (25)$$

where  $i \triangleq \sqrt{-1}$ . Then

$$\sum_{j=1}^3 (\alpha_j + i\beta_j) I_{jk} = (A + iB)(\alpha_k + i\beta_k) \quad (k = 1, 2, 3) \quad (26)$$

(25) (22) (24) (25)

and, after separating the real and imaginary parts of this equation, one has

$$\sum_{j=1}^3 \alpha_j I_{jk} = A\alpha_k - B\beta_k \quad (k = 1, 2, 3) \quad (27)$$

(26)

$$\sum_{j=1}^3 \beta_j I_{jk} = B\alpha_k + A\beta_k \quad (k = 1, 2, 3) \quad (28)$$

(26)

Multiply Eq. (28) by  $\alpha_k$  and Eq. (27) by  $\beta_k$ , and subtract, obtaining

$$\sum_{j=1}^3 \alpha_k \beta_j I_{jk} - \sum_{j=1}^3 \alpha_j \beta_k I_{jk} = B(\alpha_k^2 + \beta_k^2) \quad (k = 1, 2, 3) \quad (29)$$

and add these three equations, which yields

$$\sum_{k=1}^3 \sum_{j=1}^3 \alpha_k \beta_j I_{jk} - \sum_{k=1}^3 \sum_{j=1}^3 \alpha_j \beta_k I_{jk} = B \sum_{k=1}^3 (\alpha_k^2 + \beta_k^2) \quad (30)$$



Now,

$$\sum_{k=1}^3 \sum_{j=1}^3 \alpha_k \beta_j I_{jk} \equiv \sum_{k=1}^3 \sum_{j=1}^3 \alpha_j \beta_k I_{kj} \underset{(3.3.4)}{=} \sum_{k=1}^3 \sum_{j=1}^3 \alpha_j \beta_k I_{jk} \quad (31)$$

Consequently, the left-hand member of Eq. (30) is equal to zero, and Eq. (30) reduces to

$$B \sum_{k=1}^3 (\alpha_k^2 + \beta_k^2) = 0 \quad (32)$$

The quantities  $\alpha_k$  and  $\beta_k$  ( $k = 1, 2, 3$ ) cannot all vanish, for this would mean [see Eqs. (25)] that  $z_1, z_2, z_3$  all vanish, which is ruled out by Eq. (23). Hence, the only way Eq. (32) can be satisfied is for  $B$  to be equal to zero, and Eq. (24) thus shows that  $I_z$  is real.

Suppose that two roots of Eq. (7), say,  $I_x$  and  $I_y$ , are distinct from each other. Then, with self-explanatory notation, we can write

$$\mathbf{I}_x = I_x \mathbf{n}_x \quad (33)$$

$$\mathbf{I}_y = I_y \mathbf{n}_y \quad (34)$$

whereupon dot-multiplication of Eq. (33) with  $\mathbf{n}_y$ , Eq. (34) with  $\mathbf{n}_x$ , and subtraction of the resulting equations produces

$$\mathbf{I}_x \cdot \mathbf{n}_y - \mathbf{I}_y \cdot \mathbf{n}_x = (I_x - I_y) \mathbf{n}_x \cdot \mathbf{n}_y \quad (35)$$

or, in view of Eq. (3.3.2),

$$I_{xy} - I_{yx} \underset{(3.3.4)}{=} 0 \underset{(3.5)}{=} (I_x - I_y) \mathbf{n}_x \cdot \mathbf{n}_y \quad (36)$$

Since  $I_x$  differs from  $I_y$  by hypothesis, it follows that

$$\mathbf{n}_x \cdot \mathbf{n}_y = 0 \quad (37)$$

which proves that, whenever two principal moments of inertia of  $S$  for  $O$  are unequal, the corresponding principal axes of  $S$  for  $O$  are perpendicular to each other. Consequently, when Eq. (7) has three distinct roots, a unique principal axis of  $S$  for  $O$  corresponds to each root, and these three principal axes of  $S$  for  $O$  are mutually perpendicular. What remains to be done, is to deal with the matter of repeated roots of Eq. (7).

Let  $I_c$  denote one of the roots of Eq. (7), let  $\mathbf{n}_c$  be a unit vector parallel to the associated principal axis of  $S$  for  $O$ , and suppose that the remaining two roots of Eq. (7) are equal to each other (and possibly to  $I_c$ ). Let  $\mathbf{n}_a$  and  $\mathbf{n}_b$  be any unit vectors perpendicular to  $\mathbf{n}_c$  and to each other, and note that, in accordance with Eq. (2),

$$I_{ac} \underset{(3.3.4)}{=} I_{ca} = 0 \quad I_{bc} \underset{(3.3.4)}{=} I_{cb} = 0 \quad (38)$$

In Eq. (7), the subscripts 1, 2, and 3 may be replaced with  $a$ ,  $b$ , and  $c$ , respectively. When this is done and Eqs. (38) are used, Eq. (7) reduces to

$$(I_c - I_z)[I_z^2 - (I_a + I_b)I_z + I_a I_b - I_{ab}^2] = 0 \quad (39)$$

which has the three roots

$$I_z = \frac{I_a + I_b}{2} \pm \left[ \left( \frac{I_a - I_b}{2} \right)^2 + I_{ab}^2 \right]^{1/2}, \quad I_c \quad (40)$$

The first two of these can be equal to each other only if

$$\left( \frac{I_a - I_b}{2} \right)^2 + I_{ab}^2 = 0 \quad (41)$$

which is possible only if

$$I_a = I_b \quad (42)$$

and

$$I_{ab} = 0 \quad (43)$$

But Eqs. (40) shows that, under these circumstances,  $I_a$  is one of the values of  $I_z$ ; that is,  $I_a$  is a principal moment of inertia of  $S$  for  $O$ , and  $\mathbf{n}_a$  is thus parallel to a principal axis of  $S$  for  $O$ . Since  $\mathbf{n}_a$  was restricted only to the extent of being required to be perpendicular to  $\mathbf{n}_c$ , this means that every line passing through  $O$  and perpendicular to  $\mathbf{n}_c$  is a principal axis of  $S$  for  $O$ ; it follows from this that, when Eq. (7) has three equal roots, every line passing through  $O$  is a principal axis of  $S$  for  $O$ .

The validity of Eqs. (12) is established by Eqs. (40). Finally, Eq. (11) may be derived from Eqs. (9) by writing the first of these with the subscripts 1, 2, and 3 replaced with  $a$ ,  $b$ , and  $c$ , respectively, and with  $I_{ca}$  set equal to zero in accordance with Eqs. (38), which gives

$$z_a I_a + z_b I_{ab} = I_z z_a \quad (44)$$

where  $z_a$  and  $z_b$  are then the  $\mathbf{n}_a$  and  $\mathbf{n}_b$  measure numbers of a unit vector  $\mathbf{n}_z$  that is parallel to one of the principal axes of  $S$  for  $O$  associated with the first two values of  $I_z$  in Eqs. (40). The unit vectors  $\mathbf{n}_a$ ,  $\mathbf{n}_b$ , and  $\mathbf{n}_z$  are shown in Fig. 3.8.1, as is the angle  $\theta$ , which is seen to satisfy the equation

$$\tan \theta = \frac{z_b}{z_a} \quad (45)$$

Now,

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2(z_b/z_a)}{(45) 1 - (z_b/z_a)^2} \quad (46)$$

and

$$\frac{z_b}{z_a} = \frac{I_z - I_a}{I_{ab}} = -\frac{1}{I_{ab}} \left\{ \frac{I_a - I_b}{2} \pm \left[ \left( \frac{I_a - I_b}{2} \right)^2 + I_{ab}^2 \right]^{1/2} \right\} \quad (47)$$

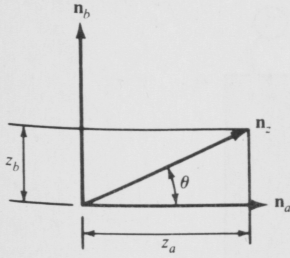


Figure 3.8.1

so that

$$1 - \left( \frac{z_b}{z_a} \right)^2 = - \frac{I_a - I_b}{I_{ab}^2} \left\{ \frac{I_a - I_b}{2} \pm \left[ \left( \frac{I_a - I_b}{2} \right)^2 + I_{ab}^2 \right]^{1/2} \right\} \quad (48)$$

Thus, Eq. (11) is obtained by substituting from Eqs. (47) and (48) into Eq. (46).

**Example** A uniform rectangular plate  $B$  of mass  $m$  has the dimensions shown in Fig. 3.8.2. Two sets of principal moments of inertia and associated principal axes of  $B$  are to be found, namely, the principal moments of inertia  $I_x, I_y, I_z$  of  $B$  for  $O$  and the associated principal axes  $X, Y, Z$ , and the principal moments of inertia  $I'_x, I'_y, I'_z$  of  $B$  for  $O'$  and the associated principal axes  $X', Y', Z'$ , where  $O'$  is a point situated at a distance  $3L$  from  $O$  on a line passing through  $O$  and parallel to  $\mathbf{n}_3$ , as shown in Fig. 3.8.2, and  $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$ .

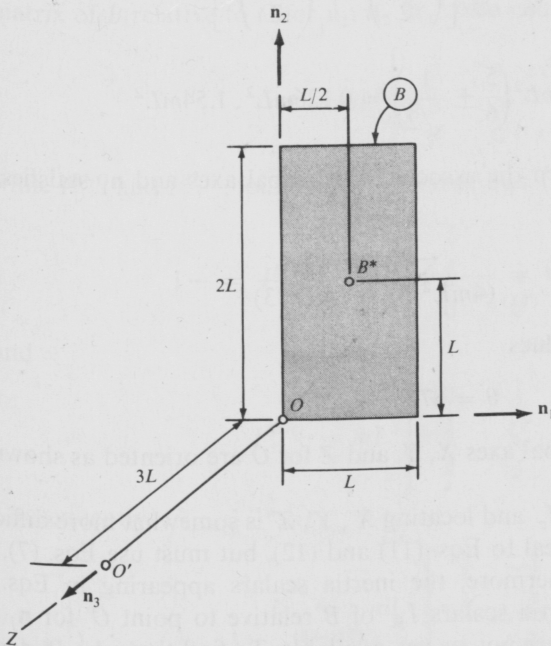


Figure 3.8.2

**Table 3.8.1**

$I_{jk}$	1	2	3
1	$4mL^2/3$	$-mL^2/2$	0
2	$-mL^2/2$	$mL^2/3$	0
3	0	0	$5mL^2/3$

When, as in the example in Sec. 3.3,  $B$  is modeled as matter distributed over a rectangular surface, this surface is a principal plane of  $B$  for  $O$ , the line  $Z$  passing through  $O$  and parallel to  $\mathbf{n}_3$  is a principal axis of  $S$  for  $O$ , and the corresponding principal moment of inertia of  $B$  for  $O$ ,  $I_z$ , is equal to the moment of inertia of  $B$  about  $Z$ . Referring to Table 3.8.1, where the inertia scalars  $I_{jk}$  of  $B$  relative to  $O$  for  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ) are recorded in accordance with Table 3.4.1, one thus finds that

$$I_z = I_{33} = \frac{5mL^2}{3} = 1.67mL^2 \quad (49)$$

and the remaining principal moments of inertia of  $B$  for  $O$ ,  $I_x$  and  $I_y$ , found with the aid of Eq. (12), where the subscripts  $a$  and  $b$  may be replaced with 1 and 2, respectively, are given by

$$\begin{aligned} I_x, I_y &= \frac{5mL^2}{6} \pm \left[ \left( \frac{mL^2}{2} \right)^2 + \left( \frac{mL^2}{2} \right)^2 \right]^{1/2} \\ &= mL^2 \left( \frac{5}{6} \pm \frac{1}{\sqrt{2}} \right) = 0.126mL^2, 1.54mL^2 \end{aligned} \quad (50)$$

while the angle  $\theta$  between the associated principal axes and  $\mathbf{n}_1$  satisfies the equation

$$\tan 2\theta = \frac{-2(mL^2/2)}{{}_{(11)}(4mL^2/3) - (mL^2/3)} = -1 \quad (51)$$

so that  $\theta$  can have the values

$$\theta = 67.5^\circ, -22.5^\circ \quad (52)$$

Consequently, the principal axes  $X$ ,  $Y$ , and  $Z$  for  $O$  are oriented as shown in Fig. 3.8.3.

Determining  $I_x'$ ,  $I_y'$ ,  $I_z'$  and locating  $X'$ ,  $Y'$ ,  $Z'$  is somewhat more difficult, for here one cannot appeal to Eqs. (11) and (12), but must use Eqs. (7), (9), and (10), instead. Furthermore, the inertia scalars appearing in Eqs. (7) and (9) must be the inertia scalars  $I_{jk}'$  of  $B$  relative to point  $O'$  for  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ), which are not, as yet, available. To find them, let  $I^*$  denote

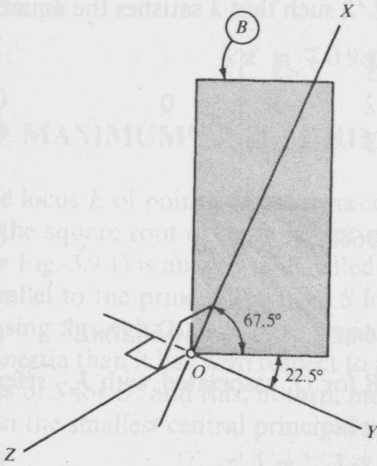


Figure 3.8.3

the inertia matrix of  $B$  relative to  $B^*$ , the mass center of  $B$ , for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , and refer to Eq. (3.6.2) to write

$$I^* = I - I^{B^*/O} \quad (53)$$

$$\begin{aligned} I' &= I^* + I^{B^*/O'} \\ &= I - I^{B^*/O} + I^{B^*/O'} \end{aligned} \quad (54)$$

(53)

where  $I'$  is the inertia matrix of  $B$  relative to  $O'$  for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , and  $I$ , the inertia matrix of  $B$  relative to  $O$  for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ , then can be written (see Table 3.8.1)

$$I = mL^2 \begin{bmatrix} \frac{4}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} \end{bmatrix} \quad (55)$$

while  $I^{B^*/O}$  and  $I^{B^*/O'}$ , found with the aid of Eqs. (3.3.3), are given by

$$I^{B^*/O} = mL^2 \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{5}{4} \end{bmatrix} \quad (56)$$

and

$$I^{B^*/O'} = mL^2 \begin{bmatrix} 10 & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{37}{4} & 3 \\ \frac{3}{2} & 3 & \frac{5}{4} \end{bmatrix} \quad (57)$$

Consequently,

$$I' = mL^2 \begin{bmatrix} \frac{31}{3} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{28}{3} & 3 \\ \frac{3}{2} & 3 & \frac{5}{3} \end{bmatrix} \quad (58)$$

(54-57)

The eigenvalues of  $I'$  are values of  $mL^2\lambda$  such that  $\lambda$  satisfies the equation

$$\begin{vmatrix} \frac{31}{3} - \lambda & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{28}{3} - \lambda & 3 \\ \frac{3}{2} & 3 & \frac{5}{3} - \lambda \end{vmatrix} = 0 \quad (7,58) \quad (59)$$

which is the case for

$$\lambda = 0.381131, 10.3665, 10.5857 \quad (60)$$

Hence,

$$I_x' = 0.381131mL^2 \quad I_y' = 10.3665mL^2 \quad I_z' = 10.5857mL^2 \quad (61)$$

To locate  $X'$ , the principal axis of  $B$  for  $O'$  associated with  $I_x'$ , refer to Eqs. (9) to write

$$z_1 I_{11}' + z_2 I_{21}' + z_3 I_{31}' = I_x' z_1 \quad (62)$$

$$z_1 I_{12}' + z_2 I_{22}' + z_3 I_{32}' = I_x' z_2 \quad (63)$$

or, by reference to Eqs. (58) and (61),

$$z_1 \left( \frac{31}{3} - 0.381131 \right) + z_2 \left( -\frac{1}{2} \right) = z_3 \left( -\frac{3}{2} \right) \quad (62) \quad (64)$$

$$z_1 \left( -\frac{1}{2} \right) + z_2 \left( \frac{28}{3} - 0.381131 \right) = z_3 (-3) \quad (63) \quad (65)$$

Solution of these two equations for  $z_1$  and  $z_2$  yields

$$z_1 = -0.1680z_3 \quad z_2 = -0.3445z_3 \quad (66)$$

and substitution into Eq. (10) shows that  $z_3$  must satisfy the equation

$$[(-0.1680)^2 + (-0.3445)^2 + 1]z_3^2 = 1 \quad (67)$$

which is satisfied by

$$z_3 = \pm 0.9338 \quad (68)$$

Hence,

$$z_1 = \mp 0.1569 \quad z_2 = \mp 0.3217 \quad (66,68) \quad (69)$$

and  $X'$  is parallel to the unit vector  $\mathbf{x}'$  defined as

$$\mathbf{x}' \triangleq \underset{(69)}{\mp 0.157\mathbf{n}_1} \underset{(69)}{\mp 0.322\mathbf{n}_2} \underset{(68)}{\pm 0.934\mathbf{n}_3} \quad (70)$$

Similarly, to locate  $Y'$  and  $Z'$ , one needs only to replace 0.381131 with 10.3665 and 10.5857, respectively [see Eqs. (61)]. in Eqs. (64) and (65), solve the resulting equations for  $z_1$  and  $z_2$  in terms of  $z_3$ , and then use Eq. (10) to find that  $Y'$  and  $Z'$  are parallel to the unit vectors  $\mathbf{y}'$  and  $\mathbf{z}'$  defined as

$$\mathbf{y}' \triangleq \mp 0.0724\mathbf{n}_1 \pm 0.947\mathbf{n}_2 \pm 0.314\mathbf{n}_3 \quad (71)$$

and

$$\mathbf{z}' = \mp 0.985\mathbf{n}_1 \mp 0.0185\mathbf{n}_2 \mp 0.172\mathbf{n}_3 \quad (72)$$

### 3.9 MAXIMUM AND MINIMUM MOMENTS OF INERTIA

The locus  $E$  of points whose distance  $R$  from a point  $O$  is inversely proportional to the square root of the moment of inertia of a set  $S$  of particles about line  $OP$  (see Fig. 3.9.1) is an ellipsoid, called an *inertia ellipsoid* of  $S$  for  $O$ , whose axes are parallel to the principal axes of  $S$  for  $O$  (see Sec. 3.8). It follows that, of all lines passing through  $O$ , those with respect to which  $S$  has a larger, or smaller, moment of inertia than it has with respect to all other lines passing through  $O$  are principal axes of  $S$  for  $O$ ; and this, in turn, means that no moment of inertia of  $S$  is smaller than the smallest central principal moment of inertia of  $S$ .

**Derivations** In Fig. 3.9.1,  $X, Y, Z$  designate mutually perpendicular coordinate axes passing through  $O$ , chosen in such a way that each is a principal axis of  $S$  for  $O$ ;  $x, y, z$  are the coordinates of  $P$ ;  $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$  are unit vectors parallel to  $X, Y, Z$ , respectively; and  $\mathbf{n}_a$  is a unit vector parallel to  $OP$ . It is to be shown that  $x, y, z$  satisfy the equation of an ellipsoid whenever

$$R = \lambda I_a^{-1/2} \quad (1)$$

where  $I_a$  is the moment of inertia of  $S$  with respect to line  $OP$ , and  $\lambda$  is any constant.

Let

$$a_x \triangleq \mathbf{n}_a \cdot \mathbf{n}_x \quad a_y \triangleq \mathbf{n}_a \cdot \mathbf{n}_y \quad a_z \triangleq \mathbf{n}_a \cdot \mathbf{n}_z \quad (2)$$

so that

$$\mathbf{n}_a = a_x \mathbf{n}_x + a_y \mathbf{n}_y + a_z \mathbf{n}_z \quad (3)$$

and note that the position vector from  $O$  to  $P$  can be expressed both as  $R\mathbf{n}_a$  and as  $x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z$ , so that, in view of Eq. (3),

$$a_x = \frac{x}{R} \quad a_y = \frac{y}{R} \quad a_z = \frac{z}{R} \quad (4)$$

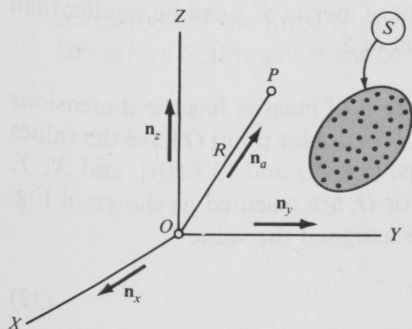


Figure 3.9.1

Also,

$$I_a = a_x^2 I_x + a_y^2 I_y + a_z^2 I_z \quad (5)$$

(3.8.4)

where  $I_x, I_y, I_z$  are the principal moments of inertia of  $S$  for  $O$ . Hence,

$$I_a = \frac{1}{R^2} (x^2 I_x + y^2 I_y + z^2 I_z) \quad (6)$$

(5.4)

and elimination of  $R$  by reference to Eq. (1) leads to

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \quad (7)$$

where  $\alpha, \beta, \gamma$  are defined as

$$\alpha \triangleq \lambda I_x^{-1/2} \quad \beta \triangleq \lambda I_y^{-1/2} \quad \gamma \triangleq \lambda I_z^{-1/2} \quad (8)$$

Equation (7) is the equation of an ellipsoid whose axes are  $X, Y, Z$ , that is, the principal axes of  $S$  for  $O$ , and whose semidiameters have lengths  $\alpha, \beta, \gamma$  given by Eqs. (8). Now, the naming of the axes always can be accomplished in such a way that  $I_z \geq I_y \geq I_x$  or, in view of Eqs. (8), that  $\gamma \leq \beta \leq \alpha$ , and the distance  $R$  from the center  $O$  to any point  $P$  of an ellipsoid is never smaller than the smallest semidiameter, and never larger than the largest semidiameter, which then means that

$$\gamma \leq R \leq \alpha \quad (9)$$

Using Eqs. (1) and (8) to eliminate  $R, \alpha$ , and  $\gamma$  gives

$$I_z^{-1/2} \leq I_a^{-1/2} \leq I_x^{-1/2} \quad (10)$$

or, equivalently,

$$I_z \geq I_a \geq I_x \quad (11)$$

which shows that the moment of inertia of  $S$  about a line that passes through  $O$  and is not a principal axis of  $S$  for  $O$  cannot be smaller than the smallest, or larger than the largest, principal moment of inertia of  $S$  for  $O$ . Finally, the moment of inertia of  $S$  about a line that does not pass through the mass center of  $S$  always exceeds the moment of inertia of  $S$  about a parallel line that does pass through the mass center [see Eq. (3.6.5)]. Hence, no moment of inertia of  $S$  can be smaller than the smallest central principal moment of inertia of  $S$ .

**Example** When a uniform rectangular plate  $B$  of mass  $m$  has the dimensions shown in Fig. 3.8.2, its principal moments of inertia for point  $O$  have the values  $0.126mL^2$ ,  $1.54mL^2$ , and  $1.67mL^2$  [see Eqs. (3.8.49) and (3.8.50)], and  $X, Y$ , and  $Z$ , the respective principal axes of  $B$  for  $O$ , are oriented as shown in Fig. 3.8.3. Hence, if the constant  $\lambda$  in Eqs. (8) is assigned the value

$$\lambda = \frac{L^2 \sqrt{m}}{2} \quad (12)$$



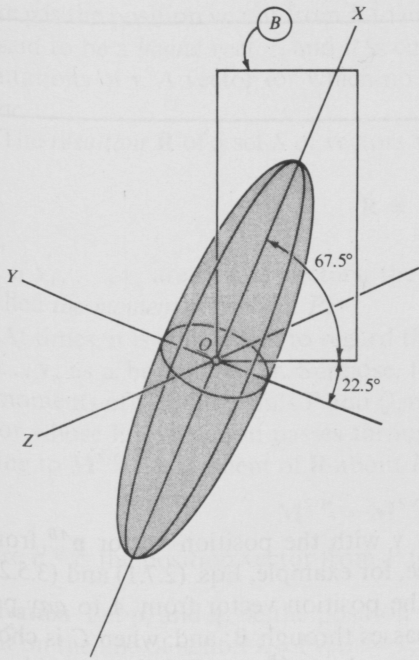


Figure 3.9.2

then the principal semidiameters of the associated inertia ellipsoid of  $B$  for  $O$  have lengths  $\alpha$ ,  $\beta$ , and  $\gamma$  given, in accordance with Eqs. (8), by

$$\alpha = \frac{L}{2\sqrt{0.126}} = 1.4L \quad \beta = 0.40L \quad \gamma = 0.39L \quad (13)$$

and the ellipsoid appears as shown in Fig. 3.9.2. Furthermore, the moment of inertia of  $B$  about line  $X$  is smaller than the moment of inertia of  $B$  about any other line passing through point  $O$ . This fact is of practical interest in the following situation. Suppose that a shaft  $S$  is fixed in a reference frame  $A$ , and it is desired to mount  $B$  on  $S$  in such a way that the axis of  $S$  passes through  $O$  while the magnitude of a force applied to  $B$  normal to the plane of  $B$  in order to impart to  $B$  any angular acceleration in  $A$  is as small as possible. Since the magnitude of the force is proportional to the moment of inertia of  $B$  about the axis of  $S$ , this axis must be made to coincide with  $X$ .

**GENERALIZED FORCES**

The necessity to cross-multiply a vector  $\mathbf{v}$  with the position vector  $\mathbf{p}^{AB}$  from a point  $A$  to a point  $B$  arises frequently [see, for example, Eqs. (2.7.1) and (3.5.27)]. Now,  $\mathbf{p}^{AB} \times \mathbf{v} = \mathbf{p}^{AC} \times \mathbf{v}$ , where  $\mathbf{p}^{AC}$  is the position vector from  $A$  to *any* point  $C$  of the line  $L$  that is parallel to  $\mathbf{v}$  and passes through  $B$ ; and, when  $C$  is chosen properly, it may be easier to evaluate  $\mathbf{p}^{AC} \times \mathbf{v}$  than  $\mathbf{p}^{AB} \times \mathbf{v}$ . This fact provides the motivation for introducing the concepts of “bound” vectors and “moments” of such vectors as in Sec. 4.1. The terms “couple” and “torque,” which have to do with special sets of bound vectors, are defined in Sec. 4.2, and the concepts of “equivalence” and “replacement,” each of which involves two sets of bound vectors, are discussed in Sec. 4.3. This material then is used throughout the rest of the chapter to facilitate the forming of expressions for quantities that play a pre-eminent role in connection with dynamical equations of motion, namely, two kinds of generalized forces. Sections 4.4–4.10 deal with generalized active forces, which come into play whenever the particles of a system are subject to the actions of contact and/or distance forces. Generalized inertia forces, which depend on both the motion and the mass distribution of a system, are discussed in Sec. 4.11. Mastery of the material brings one into position to formulate dynamical equations for any system possessing a finite number of degrees of freedom, as may be ascertained by reading Sec. 6.1.

**4.1 MOMENT ABOUT A POINT, BOUND VECTORS, RESULTANT**

Of the infinitely many lines that are parallel to every vector  $\mathbf{v}$ , a particular one, say,  $L$ , called the *line of action* of  $\mathbf{v}$ , must be selected before  $\mathbf{M}$ , the *moment* of  $\mathbf{v}$  about a point  $P$ , can be evaluated, for  $\mathbf{M}$  is defined as

$$\mathbf{M} \triangleq \mathbf{p} \times \mathbf{v} \quad (1)$$

where  $\mathbf{p}$  is the position vector from  $P$  to *any* point on  $L$ . Once  $L$  has been specified,  $\mathbf{v}$  is said to be a *bound vector*, and it is customary to show  $\mathbf{v}$  on  $L$  in pictorial representations of  $\mathbf{v}$ . A vector for which no line of action is specified is called a *free vector*.

The *resultant*  $\mathbf{R}$  of a set  $S$  of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_v$  is defined as

$$\mathbf{R} \triangleq \sum_{i=1}^v \mathbf{v}_i \quad (2)$$

and, if  $\mathbf{v}_1, \dots, \mathbf{v}_v$  are bound vectors, the sum of their moments about a point  $P$  is called the *moment of  $S$  about  $P$* .

At times, it is convenient to regard the resultant  $\mathbf{R}$  of a set  $S$  of bound vectors  $\mathbf{v}_1, \dots, \mathbf{v}_v$  as a bound vector. Suppose, for example, that  $\mathbf{M}^{S/P}$  and  $\mathbf{M}^{S/Q}$  denote the moments of  $S$  about points  $P$  and  $Q$ , respectively, and  $\mathbf{R}$  is regarded as a bound vector whose line of action passes through  $Q$ . Then one can find  $\mathbf{M}^{S/P}$  simply by adding to  $\mathbf{M}^{S/Q}$  the moment of  $\mathbf{R}$  about  $P$ , for

$$\mathbf{M}^{S/P} = \mathbf{M}^{S/Q} + \mathbf{r}^{PQ} \times \mathbf{R} \quad (3)$$

where  $\mathbf{r}^{PQ}$  is the position vector from  $P$  to  $Q$ .

**Derivation** Let  $\mathbf{p}_i$  and  $\mathbf{q}_i$  be the position vectors from  $P$  and  $Q$ , respectively, to a point on the line of action  $L_i$  of  $\mathbf{v}_i$  ( $i = 1, \dots, v$ ), and let  $\mathbf{r}^{PQ}$  be the position vector from  $P$  to  $Q$ , as shown in Fig. 4.1.1. Then, by definition,

$$\mathbf{M}^{S/P} = \sum_{i=1}^v \mathbf{p}_i \times \mathbf{v}_i \quad (4)$$

and

$$\mathbf{M}^{S/Q} = \sum_{i=1}^v \mathbf{q}_i \times \mathbf{v}_i \quad (5)$$

Now (see Fig. 4.1.1),

$$\mathbf{p}_i = \mathbf{r}^{PQ} + \mathbf{q}_i \quad (i = 1, \dots, v) \quad (6)$$

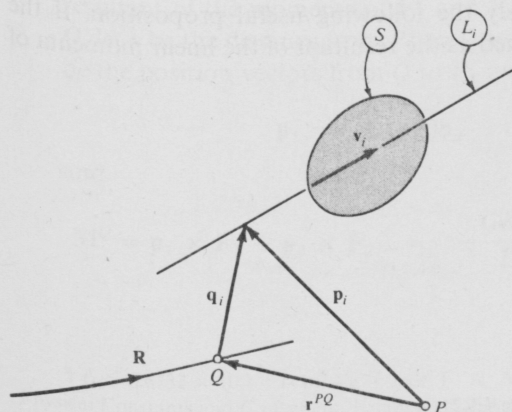


Figure 4.1.1

Hence,

$$\begin{aligned} \mathbf{M}^{S/P} &= \sum_{(4) \ i=1}^v (\mathbf{r}^{PQ} + \mathbf{q}_i) \times \mathbf{v}_i = \mathbf{r}^{PQ} \times \sum_{i=1}^v \mathbf{v}_i + \sum_{i=1}^v \mathbf{q}_i \times \mathbf{v}_i \\ &= \mathbf{r}^{PQ} \times \mathbf{R} + \mathbf{M}^{S/Q} \end{aligned} \quad (7)$$

(2)                      (5)

This establishes the validity of Eq. (3). If  $\mathbf{R}$  is regarded as a bound vector whose line of action passes through  $Q$ , then, in accordance with Eq. (1),  $\mathbf{r}^{PQ} \times \mathbf{R}$  is the moment of  $\mathbf{R}$  about  $P$ .

**Example** In Fig. 4.1.2,  $P$  and  $Q$  are two points of a rigid body  $B$  that is moving in a reference frame  $A$ . If the angular velocity  ${}^A\boldsymbol{\omega}^B$  of  $B$  in  $A$  is regarded as a bound vector whose line of action passes through  $Q$ , then  $\mathbf{M}$ , the moment of  ${}^A\boldsymbol{\omega}^B$  about point  $P$ , is given by

$$\mathbf{M} = -\mathbf{r} \times {}^A\boldsymbol{\omega}^B \quad (8)$$

(1)

where  $\mathbf{r}$  is the position vector from  $Q$  to  $P$ , as shown in Fig. 4.1.2. This moment vector is equal to the difference between the velocities of  $P$  and  $Q$  in  $A$ , for

$$\mathbf{M} = {}^A\boldsymbol{\omega}^B \times \mathbf{r} = \mathbf{A}\mathbf{v}^P - \mathbf{A}\mathbf{v}^Q \quad (9)$$

(8)                      (2.7.1)

If  $S$  is a set of particles  $P_1, \dots, P_v$  of masses  $m_1, \dots, m_v$ , respectively, moving in a reference frame  $A$  with velocities  $\mathbf{A}\mathbf{v}^{P_1}, \dots, \mathbf{A}\mathbf{v}^{P_v}$ , then the vector  $m_i \mathbf{A}\mathbf{v}^{P_i}$  is called the *linear momentum* of  $P_i$  in  $A$  ( $i = 1, \dots, v$ ), and, if this vector is assigned a line of action passing through  $P_i$ , the sum of the moments of the linear momenta of  $P_1, \dots, P_v$  in  $A$  about any point  $O$  is equal to the angular momentum of  $S$  relative to  $O$  in  $A$ , a conclusion that follows from Eq. (3.5.27) together with the definition of the moment of a set of bound vectors about a point.

The observation just made sheds light on the usage of the phrase “moment of momentum” in place of angular momentum, and it enables one to appeal to Eq. (3) to establish effortlessly the following useful proposition. If the linear momentum of  $S$  in  $A$ , defined as the resultant of the linear momenta of

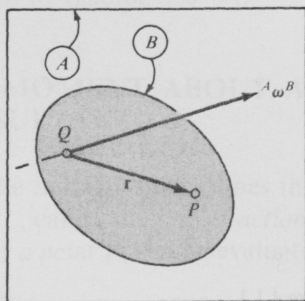


Figure 4.1.2

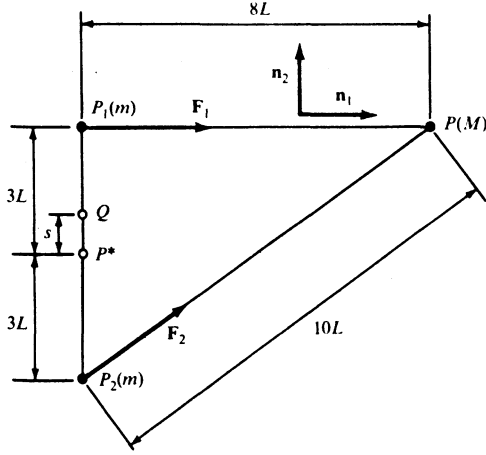


Figure 4.1.3

$P_1, \dots, P_v$  in  $A$ , is regarded as a bound vector whose line of action passes through an arbitrary point  $Q$ , then the angular momentum of  $S$  relative to any other point  $O$  is equal to the sum of the angular momentum of  $S$  relative to  $Q$  in  $A$  and the moment, about point  $O$ , of the linear momentum of  $S$  in  $A$ .

As a final example, consider the set of three particles  $P_1$ ,  $P_2$ , and  $P$  shown in Fig. 4.1.3, and let  $P_1$  and  $P_2$  each have a mass  $m$  while  $P$  has a mass  $M$ . In accordance with Newton's law of gravitation, the gravitational forces,  $\mathbf{F}_1$  and  $\mathbf{F}_2$ , exerted, respectively, on  $P_1$  and  $P_2$  by  $P$  have the lines of action shown in Fig. 4.1.3 and are given by

$$\mathbf{F}_1 = \frac{GMm}{64L^2} \mathbf{n}_1 \quad \mathbf{F}_2 = \frac{GMm}{100L^2} (0.8\mathbf{n}_1 + 0.6\mathbf{n}_2) \quad (10)$$

where  $G$  is the universal gravitational constant,<sup>†</sup> and  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are unit vectors directed as shown. Under these circumstances, there exists a point  $Q$  on line  $P^*P_1$ , where  $P^*$  is the mass center of  $P_1$  and  $P_2$ , such that  $\mathbf{M}^Q$ , the resultant of the moments of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  about  $Q$ , is equal to zero. To locate  $Q$ , let  $s$  be the distance from  $P^*$  to  $Q$ , as shown in Fig. 4.1.3, and let  $\mathbf{p}_1$  and  $\mathbf{p}_2$  be the position vectors from  $Q$  to  $P_1$  and  $P_2$ , respectively. Then

$$\mathbf{p}_1 = (3L - s)\mathbf{n}_2 \quad \mathbf{p}_2 = -(3L + s)\mathbf{n}_2 \quad (11)$$

and

$$\mathbf{M}^Q = \mathbf{p}_1 \times \mathbf{F}_1 + \mathbf{p}_2 \times \mathbf{F}_2 \underset{(11,10)}{=} -\frac{GMm}{L^2} \left[ \frac{3L - s}{64} - \frac{(3L + s)(0.8)}{100} \right] \mathbf{n}_1 \times \mathbf{n}_2 \quad (12)$$

<sup>†</sup>  $G \approx 6.6732 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$  (see E. A. Mechtly, "The International System of Units: Physical Constants and Conversion Factors," NASA SP-7012, revised, 1969).

Consequently,  $M^Q$  vanishes when

$$s = 0.968L \quad (13)$$

## 4.2 COUPLES, TORQUE

A *couple* is a set of bound vectors (see Sec. 4.1) whose resultant (see Sec. 4.1) is equal to zero. A couple consisting of only two vectors is called a *simple couple*. Hence, the vectors forming a simple couple necessarily have equal magnitudes and opposite directions.

Couples are not vectors, for a set of vectors is not a vector, any more than a set of points is a point; but there exists a unique vector, called the *torque* of the couple, that is intimately associated with a couple, namely, the moment of the couple about a point. It is unique because, as can be seen by reference to Eq. (4.1.3), a couple has the same moment about *all* points.

**Example** Four forces,  $F_1, \dots, F_4$ , have the lines of action shown in Fig. 4.2.1, and the magnitudes of  $F_1, \dots, F_4$  are proportional to the lengths of the lines  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ , respectively; that is,

$$F_1 = kan_2 \quad F_2 = k(-bn_1 + cn_3) \quad (1)$$

$$F_3 = k(bn_1 - an_2) \quad F_4 = -kc n_3 \quad (2)$$

where  $k$  is a constant and  $n_1, n_2, n_3$  are mutually perpendicular unit vectors. The forces  $F_1, \dots, F_4$  form a couple since their resultant,  $F_1 + F_2 + F_3 + F_4$ , is equal to zero. The torque  $T$  of the couple, found, for example, by adding the moments of  $F_1, \dots, F_4$  about point  $C$ , is given by (note that the moments of  $F_2$  and  $F_3$  about  $C$  are equal to zero)

$$\begin{aligned} T &= (-cn_3 + bn_1) \times F_1 + (bn_1 - an_2) \times F_4 \\ &= k(2can_1 + bcn_2 + abn_3) \end{aligned} \quad (3)$$

(1,2)

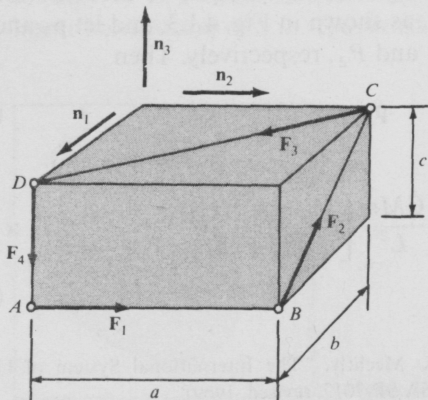


Figure 4.2.1

### 4.3 EQUIVALENCE, REPLACEMENT

Two sets of bound vectors are said to be *equivalent* when they have equal resultants and equal moments about one point; either set then is called a *replacement* of the other.

Two couples having equal torques are equivalent, for they necessarily have equal resultants (zero; see Sec. 4.2) and their respective moments about every point are equal to each other because each such moment is equal to the torque of the corresponding couple.

When two sets of bound vectors, say,  $S$  and  $S'$ , are equivalent, they have equal moments about *every* point. To see this, let  $S$  and  $S'$  have equal resultants, and let  $Q$  be *one* point about which  $S$  and  $S'$  have equal moments; then note that, in accordance with Eq. (4.1.3), the moments of  $S$  and  $S'$  about any point  $P$  other than  $Q$  depend solely on the moments of  $S$  and  $S'$  about  $Q$ , equal by hypothesis, and on the resultants of  $S$  and  $S'$ , also equal by hypothesis.

If  $S$  is any set of bound vectors while  $S'$  is a set of bound vectors consisting of a couple  $C$  of torque  $T$  together with a single bound vector  $\mathbf{v}$  whose line of action passes through a point  $P$  selected arbitrarily, then the following two requirements must be satisfied in order for  $S'$  to be a replacement of  $S$ :  $T$  is equal to the moment of  $S$  about  $P$ , and  $\mathbf{v}$  is equal to the resultant of  $S$ . For, when one of these requirements is violated,  $S$  and  $S'$  either have unequal resultants or there exists no point about which  $S$  and  $S'$  have equal moments. Conversely, satisfying both requirements guarantees the equivalence of  $S$  and  $S'$ . These facts enable one to deal in simple analytical terms with sets of bound vectors, such as contact forces exerted by one body on another, in situations in which little is known about the individual vectors of such a set.

**Example** Figure 4.3.1 is a schematic representation of a device known as Hooke's joint, described as follows. Two shafts,  $S$  and  $S'$ , are mounted in bearings,  $B$  and  $B'$ , which are fixed in a reference frame  $R$ . The axes of  $S$  and  $S'$  are parallel to unit vectors  $\mathbf{n}$  and  $\mathbf{n}'$ , respectively, and intersect at a point  $A$ . Each shaft terminates in a "yoke," and these yokes,  $Y$  and  $Y'$ , are connected to each other by a rigid cross  $C$ , one of whose arms is supported by bearings at  $D$  and  $E$  in  $Y$ , the other by bearings at  $D'$  and  $E'$  in  $Y'$ . The two arms of  $C$  have equal lengths and form a right angle with each other. Furthermore, the arm supported by  $Y$  is perpendicular to  $\mathbf{n}$ , while the one supported by  $Y'$  is perpendicular to  $\mathbf{n}'$ . Finally, circular disks  $G$  and  $G'$  having radii  $r$  and  $r'$  are attached to  $S$  and  $S'$ , respectively, and each of these is subjected to the action of a tangential force, one having a magnitude  $F$  and point of application  $P$ , the other a magnitude of  $F'$  and point of application  $P'$ .

In order for the system formed by  $G$ ,  $G'$ ,  $S$ ,  $S'$ ,  $Y$ ,  $Y'$ , and  $C$  to be in equilibrium, the ratio  $F/F'$  must be related suitably to  $r$ ,  $r'$ ,  $\mathbf{n}$ ,  $\mathbf{n}'$ ,  $\mathbf{v}$ , and  $\mathbf{v}'$ , where  $\mathbf{v}$  and  $\mathbf{v}'$  are unit vectors parallel to the arms of  $C$ , as shown in Fig. 4.3.1. To determine this relationship, one may consider the equilibrium of each of three







of action passes through point  $A$ , and the set of contact forces exerted on  $Z$  by the bearing at  $B$  is replaced with a couple of torque  $\beta$  together with a force  $\mathbf{B}$  whose line of action passes through the center of  $G$ . The vectors  $\alpha$ ,  $\mathbf{A}$ ,  $\beta$ , and  $\mathbf{B}$  are unknown, but, assuming that all bearing surfaces are so smooth that the associated contact forces have lines of action normal to the surfaces, one can conclude that  $\alpha$  must be perpendicular to  $\mathbf{v}$ , and  $\beta$  to  $\mathbf{n}$ , so that

$$\mathbf{v} \cdot \alpha = 0 \quad (1)$$

and

$$\mathbf{n} \cdot \beta = 0 \quad (2)$$

Now, when  $Z$  is in equilibrium, the sum of the moments of all forces acting on  $Z$  about the center of  $G$  is equal to zero. Hence, treating gravitational forces as negligible, one can write (see Fig. 4.3.2 for  $h$ )

$$rF\mathbf{n} + \alpha + \beta + h\mathbf{n} \times \mathbf{A} = 0 \quad (3)$$

and dot-multiplication of this equation with  $\mathbf{n}$  yields [when Eq. (2) is taken into account]

$$rF + \mathbf{n} \cdot \alpha = 0 \quad (4)$$

The body  $Z'$  and vectors  $\alpha'$ ,  $\mathbf{A}'$ ,  $\beta'$ , and  $\mathbf{B}'$  respectively analogous to  $\alpha$ ,  $\mathbf{A}$ ,  $\beta$ , and  $\mathbf{B}$  are shown in Fig. 4.3.3. Reasoning as before, one finds that

$$\mathbf{v}' \cdot \alpha' = 0 \quad (5)$$

$$\mathbf{n}' \cdot \beta' = 0 \quad (6)$$

and

$$r'F' + \mathbf{n}' \cdot \alpha' = 0 \quad (7)$$

Finally, in Fig. 4.3.4,  $C$  is shown together with vectors representing the sets of contact forces exerted on  $C$  by  $Y$  and  $Y'$ . These vectors are labeled  $-\alpha$ ,  $-\alpha'$ ,  $-\mathbf{A}$ , and  $-\mathbf{A}'$  in accordance with the law of action and reaction; that is, since to every force exerted on  $Y$  by  $C$  there corresponds a force exerted

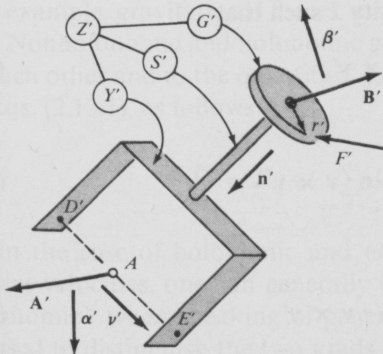


Figure 4.3.3

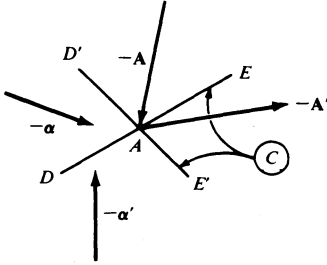


Figure 4.3.4

on  $C$  by  $Y$ , these two forces having equal magnitudes, the same line of action, but opposite senses, it follows from the definition of a replacement of a set of bound vectors that, if the set of forces exerted on  $Y$  by  $C$  is replaced with a couple of torque  $\alpha$  together with a force  $A$  whose line of action passes through point  $A$ , while the set of forces exerted on  $C$  by  $Y$  is replaced with a couple of torque  $\tilde{\alpha}$  together with a force  $\tilde{A}$  whose line of action passes through point  $A$ , then  $\tilde{\alpha} = -\alpha$  and  $\tilde{A} = -A$ . Similar considerations apply to the interaction of  $C$  with  $Y'$ .

Because  $C$  is presumed to be in equilibrium, the sum of the moments about point  $A$  of all forces acting on  $C$  may be set equal to zero, so that, in view of Fig. 4.3.4, one can write

$$-\alpha - \alpha' = 0 \quad (8)$$

Using this equation to eliminate  $\alpha'$  from Eqs. (5) and (7), one arrives at

$$\mathbf{v}' \cdot \alpha = 0 \quad (9)$$

(8) (5)

and

$$\mathbf{r}'F' - \mathbf{n}' \cdot \alpha = 0 \quad (10)$$

(8) (7)

Furthermore, Eqs. (1) and (9) show that  $\alpha$  is perpendicular both to  $\mathbf{v}$  and to  $\mathbf{v}'$ , which means that there exists a quantity  $\lambda$  such that

$$\alpha = \lambda \mathbf{v} \times \mathbf{v}' \quad (11)$$

Consequently,

$$\mathbf{r}F = -\lambda \mathbf{n} \cdot \mathbf{v} \times \mathbf{v}' \quad (12)$$

(4,11)

while

$$\mathbf{r}'F' = \lambda \mathbf{n}' \cdot \mathbf{v} \times \mathbf{v}' \quad (13)$$

(10,11)

Hence, the desired expression for  $F/F'$  is

$$\frac{F}{F'} \underset{(12.13)}{=} - \frac{r' \mathbf{n} \cdot \mathbf{v} \times \mathbf{v}'}{r \mathbf{n}' \cdot \mathbf{v} \times \mathbf{v}'} \quad (14)$$

What is most important here is to realize that it was possible to determine the relationship between  $F/F'$  and  $r, r', \mathbf{n}, \mathbf{n}', \mathbf{v}$ , and  $\mathbf{v}'$  despite the fact that relatively little is known about the contact forces exerted on  $S$  by the bearing  $B$ , on  $Y$  by  $C$ , and so forth, and that the introduction of torques together with forces having well-defined lines of action greatly facilitated the solution process.

The method just used to arrive at Eq. (14) has one major flaw, which is that it involves steps the motivation for which is not necessarily obvious, such as setting the sum of the moments of all forces acting on  $Z$  about the center of  $G$  equal to zero, dot-multiplying Eq. (3) with  $\mathbf{n}$ , and so forth. In the example in Sec. 4.5 it is shown how Eq. (14) can be obtained in a more straightforward way.

#### 4.4 GENERALIZED ACTIVE FORCES

If  $u_1, \dots, u_n$  are generalized speeds for a simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a reference frame  $A$  (see Sec. 2.13),  $p$  quantities  $\tilde{F}_1, \dots, \tilde{F}_p$ , called *nonholonomic generalized active forces* for  $S$  in  $A$ , and  $n$  quantities  $F_1, \dots, F_n$ , called *holonomic generalized active forces* for  $S$  in  $A$ , are defined as

$$\tilde{F}_r \triangleq \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i \quad (r = 1, \dots, p) \quad (1)$$

and

$$F_r \triangleq \sum_{i=1}^v \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i \quad (r = 1, \dots, n) \quad (2)$$

respectively, where  $v$  is the number of particles comprising  $S$ ,  $P_i$  is a typical particle of  $S$ ,  $\tilde{\mathbf{v}}_r^{P_i}$  and  $\mathbf{v}_r^{P_i}$  are, respectively, a nonholonomic partial velocity of  $P_i$  in  $A$  and a holonomic partial velocity of  $P_i$  in  $A$  (see Sec. 2.14), and  $\mathbf{R}_i$  is the resultant (see Sec. 4.1) of all contact forces (for example, friction forces) and distance forces (for example, gravitational forces, magnetic forces, and so forth) acting on  $P_i$ .

Nonholonomic and holonomic generalized active forces for  $S$  in  $A$  are related to each other and to the quantities  $A_{rs}$  ( $s = 1, \dots, p$ ;  $r = p + 1, \dots, n$ ) introduced in Eqs. (2.13.1), as follows:

$$\tilde{F}_r = F_r + \sum_{s=p+1}^n F_s A_{sr} \quad (r = 1, \dots, p) \quad (3)$$

As in the case of holonomic and nonholonomic partial angular velocities and partial velocities, one can generally omit the adjectives “holonomic” and “non-holonomic” when speaking of generalized forces, but the tilde notation should be used to distinguish the two kinds of generalized active forces from each other.

**Derivation** Referring to Eq. (2.14.17) to express  $\tilde{\mathbf{v}}_r^{P_i}$  ( $r = 1, \dots, p$ ) in terms of  $\mathbf{v}_s^{P_i}$  ( $s = p + 1, \dots, n$ ), one has

$$\begin{aligned}\tilde{\mathbf{F}}_r &= \sum_{(1) i=1}^v \left( \mathbf{v}_r^{P_i} + \sum_{s=p+1}^n \mathbf{v}_s^{P_i} A_{sr} \right) \cdot \mathbf{R}_i \\ &= \sum_{i=1}^v \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i + \sum_{s=p+1}^n \left( \sum_{i=1}^v \mathbf{v}_s^{P_i} \cdot \mathbf{R}_i \right) A_{sr} \quad (r = 1, \dots, p)\end{aligned}\quad (4)$$

and use of Eqs. (2) then leads immediately to Eqs. (3).

**Example** In Fig. 4.4.1,  $P_1$  and  $P_2$  designate particles of masses  $m_1$  and  $m_2$  that can slide freely in a smooth tube  $T$  and are attached to light linear springs  $\sigma_1$  and  $\sigma_2$  having spring constants  $k_1$  and  $k_2$  and “natural” lengths  $L_1$  and  $L_2$ .  $T$  is made to rotate about a fixed horizontal axis passing through one end of  $T$ , in such a way that the angle between the vertical and the axis of  $T$  is a prescribed function  $\theta(t)$  of the time  $t$ . Generalized active forces  $F_1$  and  $F_2$  associated with generalized speeds  $u_1$  and  $u_2$  defined as

$$u_r \triangleq \dot{q}_r \quad (r = 1, 2) \quad (5)$$

are to be determined for the system  $S$  formed by the two particles  $P_1$  and  $P_2$ , with  $q_1$  and  $q_2$  (see Fig. 4.4.1) designating the displacements of  $P_1$  and  $P_2$  from the positions occupied by  $P_1$  and  $P_2$  in  $T$  when  $\sigma_1$  and  $\sigma_2$  are undeformed.

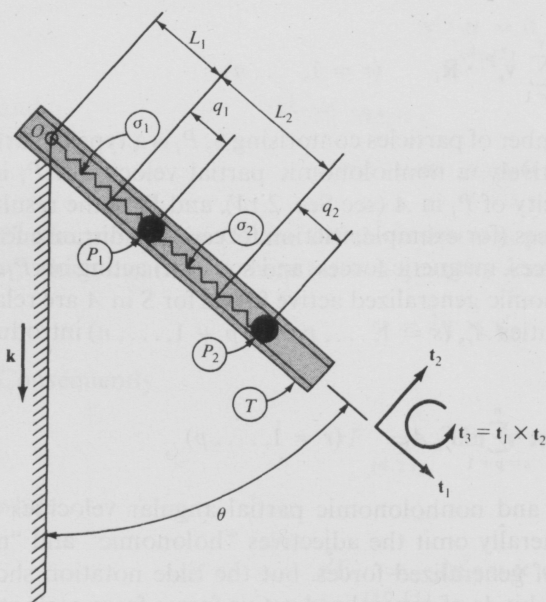


Figure 4.4.1

The velocities  $\mathbf{v}^{P_1}$  and  $\mathbf{v}^{P_2}$  of  $P_1$  and  $P_2$  can be expressed as (see Fig. 4.4.1 for the unit vectors  $\mathbf{t}_1$  and  $\mathbf{t}_2$ )

$$\mathbf{v}^{P_1} = u_1 \mathbf{t}_1 + (L_1 + q_1) \dot{\theta} \mathbf{t}_2 \quad (6)$$

(2.8.1)

$$\mathbf{v}^{P_2} = u_2 \mathbf{t}_1 + (L_1 + L_2 + q_2) \dot{\theta} \mathbf{t}_2 \quad (7)$$

(2.8.1)

$S$  is a holonomic system, and the partial velocities of  $P_1$  and  $P_2$  are

$$\mathbf{v}_1^{P_1} = \mathbf{t}_1 \quad \mathbf{v}_2^{P_1} = 0 \quad (8)$$

(6) (6)

$$\mathbf{v}_1^{P_2} = 0 \quad \mathbf{v}_2^{P_2} = \mathbf{t}_1 \quad (9)$$

(7) (7)

Contact forces are applied to  $P_1$  by  $\sigma_1$ ,  $\sigma_2$ , and  $T$ , and a gravitational force is exerted on  $P_1$  by the Earth. The forces applied to  $P_1$  by  $\sigma_1$  and  $\sigma_2$  can be written  $-k_1 q_1 \mathbf{t}_1$  and  $k_2(q_2 - q_1) \mathbf{t}_1$ , respectively, and the force exerted on  $P_1$  by  $T$  can be expressed as  $T_{12} \mathbf{t}_2 + T_{13} \mathbf{t}_3$ , where  $T_{12}$  and  $T_{13}$  are unknown scalars. No component parallel to  $\mathbf{t}_1$  is included because  $T$  is presumed to be smooth. Finally, the gravitational force exerted on  $P_1$  by the Earth is  $m_1 g \mathbf{k}$ , where  $g$  is the local gravitational acceleration and  $\mathbf{k}$  is a unit vector directed vertically downward. Hence if the gravitational force exerted on  $P_1$  by  $P_2$  is ignored,  $\mathbf{R}_1$ , the resultant of all contact and distance forces acting on  $P_1$ , is given by

$$\mathbf{R}_1 = -k_1 q_1 \mathbf{t}_1 + k_2(q_2 - q_1) \mathbf{t}_1 + T_{12} \mathbf{t}_2 + T_{13} \mathbf{t}_3 + m_1 g \mathbf{k} \quad (10)$$

and  $(F_1)_{P_1}$ , the contribution to the generalized active force  $F_1$  of all contact and distance forces acting on  $P_1$ , is

$$(F_1)_{P_1} = \mathbf{v}_1^{P_1} \cdot \mathbf{R}_1 = -k_1 q_1 + k_2(q_2 - q_1) + m_1 g \cos \theta \quad (11)$$

(2) (8.10)

Similarly, the resultant  $\mathbf{R}_2$  of all contact and distance forces acting on  $P_2$  is

$$\mathbf{R}_2 = -k_2(q_2 - q_1) \mathbf{t}_1 + T_{22} \mathbf{t}_2 + T_{23} \mathbf{t}_3 + m_2 g \mathbf{k} \quad (12)$$

so that  $(F_1)_{P_2}$ , the contribution to the generalized active force  $F_1$  of all contact and distance forces acting on  $P_2$ , is

$$(F_1)_{P_2} = \mathbf{v}_1^{P_2} \cdot \mathbf{R}_2 = 0 \quad (13)$$

(2) (9.12)

and, proceeding in the same way, one finds that  $(F_2)_{P_1}$  and  $(F_2)_{P_2}$ , the contributions to the generalized active force  $F_2$  of all contact and distance forces acting on  $P_1$  and  $P_2$ , respectively, are

$$(F_2)_{P_1} = \mathbf{v}_2^{P_1} \cdot \mathbf{R}_1 = 0 \quad (14)$$

(2) (8.10)

and

$$(F_2)_{P_2} = \underset{(2)}{\mathbf{v}_2^{P_2}} \cdot \underset{(9,12)}{\mathbf{R}_2} = -k_2(q_2 - q_1) + m_2 g \cos \theta \quad (15)$$

The desired generalized active forces are thus

$$F_1 = (F_1)_{P_1} + (F_1)_{P_2} \underset{(11,13)}{=} -k_1 q_1 + k_2(q_2 - q_1) + m_1 g \cos \theta \quad (16)$$

and

$$F_2 = (F_2)_{P_1} + (F_2)_{P_2} \underset{(14,15)}{=} -k_2(q_2 - q_1) + m_2 g \cos \theta \quad (17)$$

It is worth noting that the (unknown) contact forces exerted on  $P_1$  and  $P_2$  by  $T$  contribute nothing to the generalized active forces  $F_1$  and  $F_2$ .

## 4.5 NONCONTRIBUTING FORCES

Some forces that contribute to  $\mathbf{R}_i$  as defined in Sec. 4.4 make no contributions to the generalized active forces  $\tilde{F}_r$  ( $r = 1, \dots, p$ ) or, if  $S$  is a holonomic system, to the generalized active forces  $F_r$  ( $r = 1, \dots, n$ ). Indeed, this is the principal motivation for introducing generalized active forces. For example, the total contribution to  $\tilde{F}_r$  of all contact forces exerted on particles of  $S$  across smooth surfaces of rigid bodies vanishes; if  $B$  is a rigid body belonging to  $S$ , the total contribution to  $\tilde{F}_r$  of all contact and distance forces exerted by all particles of  $B$  on each other is equal to zero; when  $B$  rolls without slipping on a rigid body  $B'$ , the total contribution to  $\tilde{F}_r$  of all contact forces exerted on  $B$  by  $B'$  is equal to zero if  $B'$  is not a part of  $S$ , and the total contribution to  $\tilde{F}_r$  of all contact forces exerted by  $B$  and  $B'$  on each other is equal to zero if  $B'$  is a part of  $S$ .

**Derivations** Let  $\mathbf{C}$  be a contact force exerted on a particle  $P$  of  $S$  by a rigid body  $B$  across a smooth surface of  $B$ . Then, if  $\mathbf{n}$  is a unit vector normal to the surface of  $B$  at  $P$ ,

$$\mathbf{C} = C\mathbf{n} \quad (1)$$

where  $C$  is some scalar. Now consider  ${}^A\mathbf{v}^P$ , the velocity of  $P$  in  $A$ , which can be expressed as

$$\underset{(2.8.1)}{{}^A\mathbf{v}^P} = {}^A\mathbf{v}^{\bar{B}} + {}^B\mathbf{v}^P \quad (2)$$

where  ${}^A\mathbf{v}^{\bar{B}}$  is the velocity in  $A$  of the point  $\bar{B}$  of  $B$  that is in contact with  $P$ , and  ${}^B\mathbf{v}^P$  is the velocity of  $P$  in  $B$ . If  $P$  is neither to lose contact with  $B$  nor to penetrate  $B$ , then  ${}^B\mathbf{v}^P$  must be perpendicular to  $\mathbf{n}$ , and this means that

$${}^B\mathbf{v}_r^P \cdot \mathbf{n} = 0 \quad (r = 1, \dots, p) \quad (3)$$

because otherwise there can exist values of  $u_1, \dots, u_p$  such that  ${}^B\mathbf{v}^P$  is not perpendicular to  $\mathbf{n}$ . Suppose that  $B$  is part of  $S$ . Then [see Eq. (2.14.4)]

$${}^A\tilde{\mathbf{v}}_r^P = {}^A\tilde{\mathbf{v}}_r^B + {}^B\tilde{\mathbf{v}}_r^P \quad (r = 1, \dots, p) \quad (4)$$

Consequently,

$$({}^A\tilde{\mathbf{v}}_r^P - {}^A\tilde{\mathbf{v}}_r^B) \cdot \mathbf{n} = {}^B\tilde{\mathbf{v}}_r^P \cdot \mathbf{n} = 0 \quad (r = 1, \dots, p) \quad (5)$$

and the contribution to  $\tilde{\mathbf{F}}_r$  of the forces exerted on each other by  $P$  and  $B$  is [see Eq. (4.4.1)]

$${}^A\tilde{\mathbf{v}}_r^P \cdot (C\mathbf{n}) + {}^A\tilde{\mathbf{v}}_r^B \cdot (-C\mathbf{n}) = C({}^A\tilde{\mathbf{v}}_r^P - {}^A\tilde{\mathbf{v}}_r^B) \cdot \mathbf{n} = 0 \quad (r = 1, \dots, p) \quad (6)$$

Alternatively, if  $B$  is not part of  $S$ , then  $u_1, \dots, u_p$  always can be chosen in such a way that  ${}^A\tilde{\mathbf{v}}_r^B$  is independent of  $u_1, \dots, u_p$ , in which event

$${}^A\tilde{\mathbf{v}}_r^P = {}^B\tilde{\mathbf{v}}_r^P \quad (r = 1, \dots, p) \quad (7)$$

and the contribution to  $\tilde{\mathbf{F}}_r$  of the contact force exerted by  $B$  on  $P$  is [see Eq. (4.4.1)]

$${}^A\tilde{\mathbf{v}}_r^P \cdot (C\mathbf{n}) = C{}^B\tilde{\mathbf{v}}_r^P \cdot \mathbf{n} = 0 \quad (r = 1, \dots, p) \quad (8)$$

In both cases it thus follows that the total contribution to  $\tilde{\mathbf{F}}_r$  of all contact forces exerted on particles of  $S$  across smooth surfaces of rigid bodies vanishes.

In Fig. 4.5.1,  $P_i$  and  $P_j$  designate any two particles of a rigid body  $B$  belonging to  $S$ ,  $\mathbf{R}_{ij}$  is the resultant of all contact and distance forces exerted on  $P_i$  by  $P_j$ , and  $\mathbf{R}_{ji}$  is the resultant of all contact and distance forces exerted on  $P_j$  by  $P_i$ . We shall show that the total contribution of  $\mathbf{R}_{ij}$  and  $\mathbf{R}_{ji}$  to  $\tilde{\mathbf{F}}_r$  ( $r = 1, \dots, p$ ) is equal to zero, from which it follows that the total contribution to  $\tilde{\mathbf{F}}_r$  of all contact and distance forces exerted by all particles of  $B$  on each other is equal to zero.

The partial velocities  ${}^A\tilde{\mathbf{v}}_r^{P_i}$  and  ${}^A\tilde{\mathbf{v}}_r^{P_j}$  of  $P_i$  and  $P_j$  in  $A$  are related to the partial angular velocity  ${}^A\tilde{\boldsymbol{\omega}}_r^B$  of  $B$  in  $A$  by [see Eqs. (2.7.1), (2.14.3), and (2.14.4)]

$${}^A\tilde{\mathbf{v}}_r^{P_j} = {}^A\tilde{\mathbf{v}}_r^{P_i} + {}^A\tilde{\boldsymbol{\omega}}_r^B \times \mathbf{p}_{ij} \quad (r = 1, \dots, p) \quad (9)$$

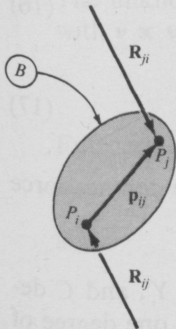


Figure 4.5.1

where  $\mathbf{p}_{ij}$  is the position vector from  $P_i$  to  $P_j$ ; and the forces  $\mathbf{R}_{ij}$  and  $\mathbf{R}_{ji}$  are related to each other by the law of action and reaction, which asserts that  $\mathbf{R}_{ij}$  and  $\mathbf{R}_{ji}$  have equal magnitudes, opposite directions, and coincident lines of action. Hence,

$$\mathbf{R}_{ji} = -\mathbf{R}_{ij} \quad (10)$$

and, since the line of action of  $\mathbf{R}_{ij}$  passes through  $P_i$ , it is parallel to  $\mathbf{p}_{ij}$ , which means that

$$\mathbf{p}_{ij} \times \mathbf{R}_{ij} = 0 \quad (11)$$

Consequently, the total contribution of  $\mathbf{R}_{ij}$  and  $\mathbf{R}_{ji}$  to  $\tilde{F}_r$  ( $r = 1, \dots, p$ ) is [see Eq. (4.4.1)]

$$\begin{aligned} {}^A\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_{ij} + {}^A\tilde{\mathbf{v}}_r^{P_j} \cdot \mathbf{R}_{ji} &= ({}^A\tilde{\mathbf{v}}_r^{P_i} - {}^A\tilde{\mathbf{v}}_r^{P_j}) \cdot \mathbf{R}_{ij} \\ &= -{}^A\tilde{\boldsymbol{\omega}}_r^B \times \mathbf{p}_{ij} \cdot \mathbf{R}_{ij} = 0 \quad (r = 1, \dots, p) \end{aligned} \quad \begin{matrix} (10) \\ (9) \end{matrix} \quad (12)$$

Finally, to deal with contact forces that come into play when  $B$  rolls on a rigid body  $B'$ , we let  $P$  be a point of  $B$ , and  $P'$  a point of  $B'$ , choosing these such that  $P$  is in rolling contact with  $P'$ , which means that

$${}^A\mathbf{v}^P = {}^A\mathbf{v}^{P'} \quad (13)$$

where  $A$  is any reference frame. Now there are two possibilities. Either  $B'$  belongs to  $S$ , in which case

$${}^A\tilde{\mathbf{v}}_r^P = {}^A\tilde{\mathbf{v}}_r^{P'} \quad (r = 1, \dots, p) \quad (14)$$

and the contribution to  $\tilde{F}_r$  of  $\mathbf{R}$  and  $\mathbf{R}'$ , the resultants of the contact forces exerted by  $B'$  on  $B$  and by  $B$  on  $B'$ , respectively, is given by

$${}^A\tilde{\mathbf{v}}_r^P \cdot \mathbf{R} + {}^A\tilde{\mathbf{v}}_r^{P'} \cdot \mathbf{R}' = {}^A\tilde{\mathbf{v}}_r^P \cdot (\mathbf{R} + \mathbf{R}') \quad (15)$$

which vanishes because, in accordance with the law of action and reaction,  $\mathbf{R} + \mathbf{R}' = 0$ . Alternatively,  $B'$  does not belong to  $S$ , in which case generalized speeds always can be chosen such that  ${}^A\mathbf{v}^{P'}$  is independent of all of them, so that

$${}^A\tilde{\mathbf{v}}_r^{P'} = 0 \quad (r = 1, \dots, p) \quad (16)$$

and, consequently,

$${}^A\tilde{\mathbf{v}}_r^P = 0 \quad (r = 1, \dots, p) \quad (17)$$

from which it follows that  ${}^A\tilde{\mathbf{v}}_r^P \cdot \mathbf{R}$ , the contribution to  $\tilde{F}_r$  of the contact force exerted on  $B$  by  $B'$  at  $P$ , is equal to zero.

**Example** The system  $H$  formed by the bodies  $G$ ,  $G'$ ,  $S$ ,  $S'$ ,  $Y$ ,  $Y'$ , and  $C$  described in the example in Sec. 4.3 is holonomic and possesses one degree of



freedom in the reference frame  $R$  shown in Fig. 4.3.1. Defining a single generalized speed  $u_1$  as

$$u_1 \triangleq {}^R\boldsymbol{\omega}^G \cdot \mathbf{n} \quad (18)$$

and letting  ${}^R\mathbf{v}_1^P$  and  ${}^R\mathbf{v}_1^{P'}$  denote the associated partial velocities of  $P$  and  $P'$ , respectively, in  $R$ , one can express the contribution of the forces of magnitudes  $F$  and  $F'$  (see Fig. 4.3.1) to the generalized active force  $F_1$  for  $H$  in  $R$  as

$$F_1 = {}^R\mathbf{v}_1^P \cdot (F\boldsymbol{\tau}) + {}^R\mathbf{v}_1^{P'} \cdot (F'\boldsymbol{\tau}') \quad (19)$$

where  $\boldsymbol{\tau}$  and  $\boldsymbol{\tau}'$  are unit vectors parallel to the tangents to  $G$  and  $G'$  at  $P$  and  $P'$ , respectively, as indicated in Fig. 4.3.1. Moreover, if it is assumed, as before, that all bearing surfaces are smooth, and if gravitational forces are once again left out of account, then  $F_1$  as given by Eq. (19) is, in fact, the total generalized active force for  $H$  in  $R$ .

In the example in Sec. 4.3, considerations of the equilibrium of each of three rigid bodies led to Eq. (4.3.14). The same result can be obtained simply by setting the generalized active force  $F_1$  equal to zero, which leads, first, to

$$\frac{F}{F'} = - \frac{{}^R\mathbf{v}_1^{P'} \cdot \boldsymbol{\tau}'}{{}^R\mathbf{v}_1^P \cdot \boldsymbol{\tau}} \quad (20)$$

Now, with self-explanatory notation,

$${}^R\boldsymbol{\omega}^{S'} = {}^R\boldsymbol{\omega}^S + {}^S\boldsymbol{\omega}^C + {}^C\boldsymbol{\omega}^{S'} \quad (21)$$

(2.4.1)

and the simple angular velocities (see Sec. 2.2) appearing in this equation can be expressed as

$${}^R\boldsymbol{\omega}^{S'} = {}^R\boldsymbol{\omega}^{S'} \mathbf{n}' \quad {}^R\boldsymbol{\omega}^S = u_1 \mathbf{n} \quad (22)$$

(18)

$${}^S\boldsymbol{\omega}^C = {}^S\boldsymbol{\omega}^C \mathbf{v} \quad {}^C\boldsymbol{\omega}^{S'} = {}^C\boldsymbol{\omega}^{S'} \mathbf{v}' \quad (23)$$

Hence,

$${}^R\boldsymbol{\omega}^{S'} \mathbf{n}' = u_1 \mathbf{n} + {}^S\boldsymbol{\omega}^C \mathbf{v} + {}^C\boldsymbol{\omega}^{S'} \mathbf{v}' \quad (24)$$

(22) (21) (22) (23) (23)

The unknowns  ${}^S\boldsymbol{\omega}^C$  and  ${}^C\boldsymbol{\omega}^{S'}$  can be eliminated by dot-multiplying Eq. (24) with  $\mathbf{v} \times \mathbf{v}'$ , which gives

$${}^R\boldsymbol{\omega}^{S'} \mathbf{n}' \cdot \mathbf{v} \times \mathbf{v}' = u_1 \mathbf{n} \cdot \mathbf{v} \times \mathbf{v}' \quad (25)$$

(24)

Furthermore, the velocities of  $P$  and  $P'$  in  $R$  are

$${}^R\mathbf{v}^P = ru_1 \boldsymbol{\tau} \quad (26)$$

and

$${}^R\mathbf{v}^{P'} = r' {}^R\boldsymbol{\omega}^{S'} \boldsymbol{\tau}' = r' u_1 \frac{\mathbf{n} \cdot \mathbf{v} \times \mathbf{v}'}{\mathbf{n}' \cdot \mathbf{v} \times \mathbf{v}'} \boldsymbol{\tau}' \quad (27)$$

(25)

so that the partial velocities  ${}^R\mathbf{v}_1^P$  and  ${}^R\mathbf{v}_1^{P'}$  are given by

$${}^R\mathbf{v}_1^P = r\boldsymbol{\tau} \quad {}^R\mathbf{v}_1^{P'} = r' \frac{\mathbf{n} \cdot \mathbf{v} \times \mathbf{v}'}{\mathbf{n}' \cdot \mathbf{v} \times \mathbf{v}'} \boldsymbol{\tau}' \quad (28)$$

(26) (27)

Substitution from Eqs. (28) into Eq. (20) leads directly to Eq. (4.3.14).

## 4.6 FORCES ACTING ON A RIGID BODY

If  $B$  is a rigid body belonging to a nonholonomic system  $S$  possessing  $p$  degrees of freedom in a reference frame  $A$  (see Sec. 2.13), and a set of contact and/or distance forces acting on  $B$  is equivalent (see Sec. 4.3) to a couple of torque  $\mathbf{T}$  (see Sec. 4.2) together with a force  $\mathbf{R}$  whose line of action passes through a point  $Q$  of  $B$ , then  $(\tilde{F}_r)_B$ , the contribution of this set of forces to the generalized active force  $\tilde{F}_r$  (see Sec. 4.4) for  $S$  in  $A$  is given by

$$(\tilde{F}_r)_B = {}^A\tilde{\omega}_r^B \cdot \mathbf{T} + {}^A\tilde{\mathbf{v}}_r^Q \cdot \mathbf{R} \quad (r = 1, \dots, p) \quad (1)$$

where  ${}^A\tilde{\omega}_r^B$  and  ${}^A\tilde{\mathbf{v}}_r^Q$  are, respectively, the  $r$ th partial angular velocity of  $B$  in  $A$  and the  $r$ th partial velocity of  $Q$  in  $A$ .

**Derivation** Let  $\mathbf{K}_1, \dots, \mathbf{K}_\beta$  be the contact and/or distance forces acting on particles  $P_1, \dots, P_\beta$  of  $B$ , and let  $\mathbf{p}_1, \dots, \mathbf{p}_\beta$  be the position vectors from  $Q$  to  $P_1, \dots, P_\beta$ , respectively. Then, by definition of equivalence, the resultant of  $\mathbf{K}_1, \dots, \mathbf{K}_\beta$  is equal to  $\mathbf{R}$ , that is,

$$\sum_{i=1}^{\beta} \mathbf{K}_i = \mathbf{R} \quad (4.1.2) \quad (2)$$

and the sum of the moments of  $\mathbf{K}_1, \dots, \mathbf{K}_\beta$  about  $Q$  is equal to  $\mathbf{T}$ , so that

$$\sum_{i=1}^{\beta} \mathbf{p}_i \times \mathbf{K}_i = \mathbf{T} \quad (4.1.1) \quad (3)$$

Also by definition, the contribution of  $\mathbf{K}_1, \dots, \mathbf{K}_\beta$  to  $\tilde{F}_r$  is

$$(\tilde{F}_r)_B = \sum_{(4.4.1) i=1}^{\beta} {}^A\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{K}_i \quad (r = 1, \dots, p) \quad (4)$$

where, with the aid of Eqs. (2.7.1), (2.14.4), and (2.14.3), one can express  ${}^A\tilde{\mathbf{v}}_r^{P_i}$  as

$${}^A\tilde{\mathbf{v}}_r^{P_i} = {}^A\tilde{\mathbf{v}}_r^Q + {}^A\tilde{\omega}_r^B \times \mathbf{p}_i \quad (r = 1, \dots, p; i = 1, \dots, \beta) \quad (5)$$

Hence,

$$\begin{aligned} (\tilde{F}_r)_B &= \sum_{(4) i=1}^{\beta} ({}^A\tilde{\mathbf{v}}_r^Q + {}^A\tilde{\omega}_r^B \times \mathbf{p}_i) \cdot \mathbf{K}_i = {}^A\tilde{\mathbf{v}}_r^Q \cdot \sum_{i=1}^{\beta} \mathbf{K}_i + {}^A\tilde{\omega}_r^B \cdot \sum_{i=1}^{\beta} \mathbf{p}_i \times \mathbf{K}_i \\ &= {}^A\tilde{\mathbf{v}}_r^Q \cdot \mathbf{R} + {}^A\tilde{\omega}_r^B \cdot \mathbf{T} \quad (r = 1, \dots, p) \end{aligned} \quad (6)$$

(2) (3)

in agreement with Eq. (1).

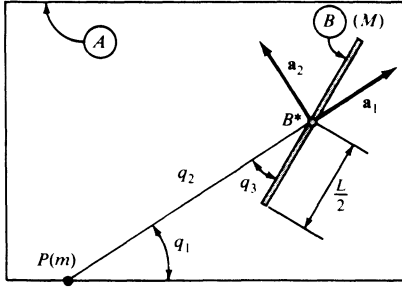


Figure 4.6.1

**Example** Figure 4.6.1 shows a uniform rod  $B$  of mass  $M$  and length  $L$ .  $B$  is free to move in a plane  $A$ , and  $P$  is a particle of mass  $m$ , fixed in  $A$ .

It can be shown† that the set of gravitational forces exerted by  $P$  on  $B$  is approximately equivalent to a couple of torque  $\mathbf{T}$  together with a force  $\mathbf{R}$  whose line of action passes through the mass center  $B^*$  of  $B$ , with  $\mathbf{T}$  and  $\mathbf{R}$  given by

$$\mathbf{T} = -\frac{GMmL^2}{8q_2^3} \sin 2q_3 \mathbf{a}_1 \times \mathbf{a}_2 \quad (7)$$

and

$$\mathbf{R} = -\frac{GMm}{q_2^2} \left\{ \mathbf{a}_1 \left[ 1 + \frac{L^2}{8q_2^2} (2 - 3 \sin^2 q_3) \right] - \mathbf{a}_2 \frac{L^2}{8q_2^2} \sin 2q_3 \right\} \quad (8)$$

where  $G$  is the universal gravitational constant,  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are unit vectors directed as shown in Fig. 4.6.1, and  $q_1, q_2, q_3$  are generalized coordinates characterizing the configuration of  $B$  in  $A$  (see Fig. 4.6.1). If generalized speeds  $u_1, u_2, u_3$  are introduced as

$$u_r \triangleq \dot{q}_r \quad (r = 1, 2, 3) \quad (9)$$

then  $\boldsymbol{\omega}$ , the angular velocity of  $B$  in  $A$ , and  $\mathbf{v}$ , the velocity of  $B^*$  in  $A$ , are given by

$$\boldsymbol{\omega} = (u_1 + u_3) \mathbf{a}_1 \times \mathbf{a}_2 \quad \mathbf{v} = u_2 \mathbf{a}_1 + u_1 q_2 \mathbf{a}_2 \quad (10)$$

so that the partial angular velocities of  $B$  in  $A$  and the partial velocities of  $B^*$  in  $A$  are

$$\boldsymbol{\omega}_1 = \mathbf{a}_1 \times \mathbf{a}_2 \quad \boldsymbol{\omega}_2 = 0 \quad \boldsymbol{\omega}_3 = \mathbf{a}_1 \times \mathbf{a}_2 \quad (11)$$

$$\mathbf{v}_1 = q_2 \mathbf{a}_2 \quad \mathbf{v}_2 = \mathbf{a}_1 \quad \mathbf{v}_3 = 0 \quad (12)$$

† T. R. Kane, P. W. Likins, and D. A. Levinson, *Spacecraft Dynamics* (McGraw-Hill, New York, 1983), Secs. 2.3, 2.6.

The contributions of the gravitational forces exerted by  $P$  on  $B$  to the generalized active forces for  $B$  in  $A$  are

$$\begin{aligned} (F_1)_B &= \underset{(1)}{\mathbf{a}_1} \times \underset{(11)}{\mathbf{a}_2} \cdot \underset{(12)}{\mathbf{T}} + q_2 \underset{(12)}{\mathbf{a}_2} \cdot \mathbf{R} \\ &= -\frac{GMmL^2}{8q_2^3} \sin 2q_3 + \frac{GMmL^2}{8q_2^3} \sin 2q_3 = 0 \end{aligned} \quad (13)$$

$$(F_2)_B \underset{(1,11,12,8)}{=} -\frac{GMm}{q_2^2} \left[ 1 + \frac{L^2}{8q_2^2} (2 - 3 \sin^2 q_3) \right] \quad (14)$$

$$(F_3)_B \underset{(1,11,12,7)}{=} -\frac{GMmL^2}{8q_2^3} \sin 2q_3 \quad (15)$$

#### 4.7 CONTRIBUTING INTERACTION FORCES

In Sec. 4.5 it was shown that certain interaction forces, that is, forces exerted by one part of a system on another, make no contributions to generalized active forces. In some situations, forces of interaction *do* contribute to generalized active forces. For example, whenever two particles of a system are not rigidly connected to each other, the gravitational forces exerted by the particles on each other can make such contributions. Bodies connected to each other by certain energy storage or energy dissipation devices furnish additional examples.

**Example** Figure 4.7.1 shows a double pendulum consisting of two rigid rods,  $A$  and  $B$ . Rod  $A$  is pinned to a fixed support, and  $A$  and  $B$  are pin-connected. Relative motion of  $A$  and  $B$  is resisted by a light torsion spring of modulus  $\sigma$  and by a viscous fluid damper with a damping constant  $\delta$ . In other words, the set of forces exerted on  $A$  by  $B$  through the spring and damper is equivalent (see Sec. 4.3) to a couple whose torque  $\mathbf{T}_A$  is given by

$$\mathbf{T}_A = (\sigma q_2 + \delta \dot{q}_2) \mathbf{n} \quad (1)$$

where  $q_2$  is the angle between  $A$  and  $B$ , as shown in Fig. 4.7.1, and the set of forces exerted by  $A$  on  $B$  through the spring and damper is equivalent to a couple whose torque  $\mathbf{T}_B$  can be written

$$\mathbf{T}_B = -\mathbf{T}_A \quad (2)$$

Suppose now that generalized speeds  $u_1$  and  $u_2$  are defined as (see Fig. 4.7.1 for  $q_1$ )

$$u_r = \dot{q}_r \quad (r = 1, 2) \quad (3)$$

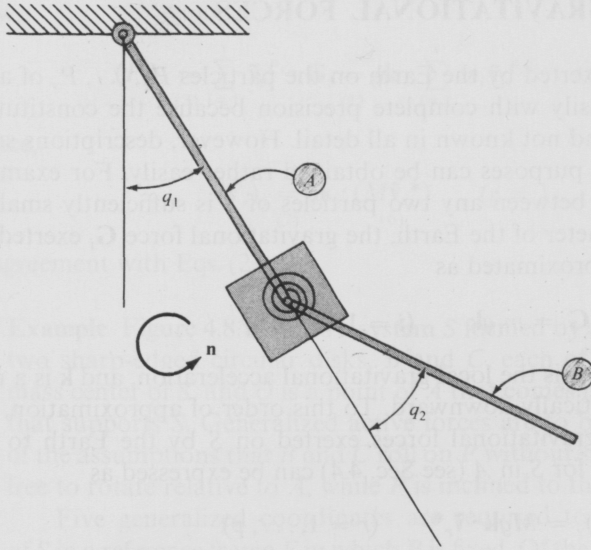


Figure 4.7.1

and let  $f_r$  ( $r = 1, 2$ ) be the contribution of the spring and damper forces to the generalized active force  $F_r$  ( $r = 1, 2$ ). Then

$$f_r = \omega_r^A \cdot \mathbf{T}_A + \omega_r^B \cdot \mathbf{T}_B \quad (r = 1, 2) \quad (4)$$

(4.6.1)

Now,

$$\omega^A = \dot{q}_1 \mathbf{n} = u_1 \mathbf{n} \quad (3)$$

$$\omega^B = (\dot{q}_1 + \dot{q}_2) \mathbf{n} = (u_1 + u_2) \mathbf{n} \quad (3)$$

so that

$$\omega_1^A = \mathbf{n} \quad \omega_2^A = 0 \quad \omega_1^B = \mathbf{n} \quad \omega_2^B = \mathbf{n} \quad (5) \quad (6) \quad (6) \quad (6) \quad (7)$$

Hence,

$$f_1 = \mathbf{n} \cdot \mathbf{T}_A + \mathbf{n} \cdot (-\mathbf{T}_A) = 0 \quad (4) \quad (7) \quad (7) \quad (2) \quad (8)$$

and

$$f_2 = 0 \cdot \mathbf{T}_A + \mathbf{n} \cdot (-\mathbf{T}_A) = -(\sigma q_2 + \delta \dot{q}_2) \quad (4) \quad (7) \quad (2) \quad (1) \quad (9)$$

Thus, the interaction forces associated with the spring and damper here contribute to  $F_2$  but not to  $F_1$ . The reader should verify that, if  $u_1$  and  $u_2$  are defined as  $u_1 = \dot{q}_1$ ,  $u_2 = \dot{q}_1 + \dot{q}_2$ , rather than as in Eqs. (3), then the spring forces and damper forces contribute to both  $F_1$  and  $F_2$ .

## 4.8 TERRESTRIAL GRAVITATIONAL FORCES

The gravitational forces exerted by the Earth on the particles  $P_1, \dots, P_v$  of a set  $S$  cannot be evaluated easily with complete precision because the constitution of the Earth is complex and not known in all detail. However, descriptions sufficiently accurate for many purposes can be obtained rather easily. For example, when the largest distance between any two particles of  $S$  is sufficiently small in comparison with the diameter of the Earth, the gravitational force  $\mathbf{G}_i$  exerted on  $P_i$  by the Earth can be approximated as

$$\mathbf{G}_i = m_i g \mathbf{k} \quad (i = 1, \dots, v) \quad (1)$$

where  $m_i$  is the mass of  $P_i$ ,  $g$  is the local gravitational acceleration, and  $\mathbf{k}$  is a unit vector locally directed vertically downward. To this order of approximation, the contribution  $(\tilde{\mathbf{F}}_r)_\gamma$  of *all* gravitational forces exerted on  $S$  by the Earth to the generalized active force  $\tilde{\mathbf{F}}_r$  for  $S$  in  $A$  (see Sec. 4.4) can be expressed as

$$(\tilde{\mathbf{F}}_r)_\gamma = M g \mathbf{k} \cdot \tilde{\mathbf{v}}_r^* \quad (r = 1, \dots, p) \quad (2)$$

where  $M$  is the total mass of  $S$  and  $\tilde{\mathbf{v}}_r^*$  is the  $r$ th partial velocity of the mass center of  $S$  in  $A$ .

As an alternative to using Eqs. (2), one can deal with  $P_1, \dots, P_v$  individually by expressing  $(\tilde{\mathbf{F}}_r)_\gamma$  as

$$(\tilde{\mathbf{F}}_r)_\gamma = \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{G}_i \quad (r = 1, \dots, p) \quad (3)$$

Whether it is more convenient to use Eqs. (2) or Eqs. (3) depends on the relative ease of finding  $\tilde{\mathbf{v}}_r^*$  ( $r = 1, \dots, p$ ), on the one hand, and  $\tilde{\mathbf{v}}_r^{P_i}$  ( $r = 1, \dots, p$ ;  $i = 1, \dots, v$ ), on the other hand.

**Derivation** The position vector  $\mathbf{p}^*$  from a point  $O$  fixed in  $A$  to the mass center of  $S$  is related to the position vectors  $\mathbf{p}_1, \dots, \mathbf{p}_v$  from  $O$  to the particles of  $S$  by

$$M \mathbf{p}^* = \sum_{(3.1.2) \ i=1}^v m_i \mathbf{p}_i \quad (4)$$

Differentiation with respect to  $t$  in  $A$  yields

$$M \mathbf{v}^* = \sum_{(2.6.1) \ (4) \ i=1}^v m_i \mathbf{v}^{P_i} \quad (5)$$

Consequently, the partial velocities  $\tilde{\mathbf{v}}_r^*$  and  $\tilde{\mathbf{v}}_r^{P_i}$  ( $r = 1, \dots, p$ ) are related to each other by [see Eq. (2.14.4)]

$$M \tilde{\mathbf{v}}_r^* = \sum_{(5) \ i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i} \quad (r = 1, \dots, p) \quad (6)$$



expressed as a linear function of  $u_1$  and  $u_2$ . For example,  $\mathbf{v}^*$ , the velocity of  $S^*$  in  $F$ , becomes (see Fig. 4.8.1 for  $\mathbf{a}_3$ )

$$\mathbf{v}^* = u_2 \mathbf{a}_2 + eu_1 \mathbf{a}_3 \quad (10)$$

(2.7.1)

and  $\mathbf{v}^Q$ , the velocity in  $F$  of the point  $Q$  of  $A$  that comes into contact with  $A$ , is

$$\mathbf{v}^Q = u_2 \mathbf{a}_2 + fu_1 \mathbf{a}_3 \quad (11)$$

(2.7.1)

The associated partial velocities are

$$\tilde{\mathbf{v}}_1^* = e\mathbf{a}_3 \quad \tilde{\mathbf{v}}_2^* = \mathbf{a}_2 \quad (12)$$

(10) (10)

$$\tilde{\mathbf{v}}_1^Q = f\mathbf{a}_3 \quad \tilde{\mathbf{v}}_2^Q = \mathbf{a}_2 \quad (13)$$

(11) (11)

The only contact force that contributes to the generalized active forces  $\tilde{F}_1$  and  $\tilde{F}_2$  is (see Sec. 4.5) the force exerted on  $A$  by  $P$  at  $Q$ . When this force is represented as  $Q_1\mathbf{a}_1 + Q_2\mathbf{a}_2 + Q_3\mathbf{a}_3$ , then  $(\tilde{F}_r)_Q$ , its contribution to  $\tilde{F}_r$ , is given by

$$(\tilde{F}_r)_Q = \tilde{\mathbf{v}}_r^Q \cdot (Q_1\mathbf{a}_1 + Q_2\mathbf{a}_2 + Q_3\mathbf{a}_3) \quad (r = 1, 2) \quad (14)$$

so that

$$(\tilde{F}_1)_Q = fQ_3 \quad (\tilde{F}_2)_Q = Q_2 \quad (15)$$

(13,14) (13,14)

Letting  $M$  denote the mass of  $S$ , we have for  $(\tilde{F}_r)_\gamma$ , the contribution to  $\tilde{F}_r$  ( $r = 1, 2$ ) of the gravitational forces exerted on  $S$  by the Earth,

$$(\tilde{F}_1)_\gamma = Mge\mathbf{a}_3 \cdot \mathbf{k} \quad (\tilde{F}_2)_\gamma = Mga_2 \cdot \mathbf{k} \quad (16)$$

(2,12) (2,12)

The values of the dot-products  $\mathbf{a}_3 \cdot \mathbf{k}$  and  $\mathbf{a}_2 \cdot \mathbf{k}$  depend on both the inclination angle  $\theta$  and the orientation of  $A$  in  $P$ , which can be characterized by introducing the angle  $q_1$  shown in Fig. 4.8.2, where  $\mathbf{n}_2$  is a horizontal unit vector perpendicular to  $\mathbf{a}_1$ , while  $\mathbf{n}_3 = \mathbf{a}_1 \times \mathbf{n}_2$ . Under these circumstances,

$$\mathbf{a}_2 = \cos q_1 \mathbf{n}_2 + \sin q_1 \mathbf{n}_3 \quad \mathbf{a}_3 = -\sin q_1 \mathbf{n}_2 + \cos q_1 \mathbf{n}_3 \quad (17)$$

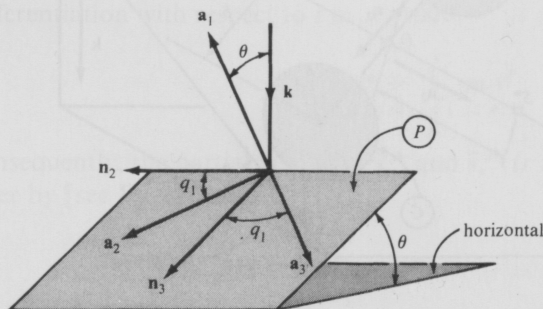


Figure 4.8.2



and

$$\mathbf{k} = -\cos \theta \mathbf{a}_1 + \sin \theta \mathbf{n}_3 \quad (18)$$

Hence,

$$(\tilde{F}_1)_\gamma = Mge \sin \theta \cos q_1 \quad (\tilde{F}_2)_\gamma = Mg \sin \theta \sin q_1 \quad (19)$$

(16-18) (16-18)

and the desired generalized active forces are

$$\tilde{F}_1 = (\tilde{F}_1)_Q + (\tilde{F}_1)_\gamma = f Q_3 + Mge \sin \theta \cos q_1 \quad (20)$$

(15,19)

and

$$\tilde{F}_2 = (\tilde{F}_2)_Q + (\tilde{F}_2)_\gamma = Q_2 + Mg \sin \theta \sin q_1 \quad (21)$$

(15,19)

The reason for using Eqs. (2) rather than Eqs. (3) to find  $(\tilde{F}_1)_\gamma$  and  $(\tilde{F}_2)_\gamma$  [see Eqs. (16)] is that it would be very laborious to deal individually with each of the particles forming  $A$ ,  $B$ , and  $C$ . By way of contrast, consider once again the contributions of gravitational forces to the generalized active forces  $F_1$  and  $F_2$  in the example in Sec. 4.4. Since the partial velocities of  $P_1$  and  $P_2$  are available in Eqs. (4.4.8) and (4.4.9),  $(F_1)_\gamma$  and  $(F_2)_\gamma$  are formed easily as

$$(F_1)_\gamma = \mathbf{v}_1^{P_1} \cdot (m_1 g \mathbf{k}) + \mathbf{v}_1^{P_2} \cdot (m_2 g \mathbf{k}) = m_1 g \cos \theta + 0 \quad (22)$$

(3) (4.4.8) (4.4.9)

$$(F_2)_\gamma = \mathbf{v}_2^{P_1} \cdot (m_1 g \mathbf{k}) + \mathbf{v}_2^{P_2} \cdot (m_2 g \mathbf{k}) = 0 + m_2 g \cos \theta \quad (23)$$

(3) (4.4.8) (4.4.9)

whereas, to use Eqs. (2), one must first locate the mass center  $S^*$  of  $P_1$  and  $P_2$ , determine its velocity, and use this to form partial velocities. Specifically, letting  $\mathbf{p}^*$  be the position vector from point  $O$  in Fig. 4.4.1 to  $S^*$ , one has

$$\mathbf{p}^* = \frac{m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)}{m_1 + m_2} \mathbf{t}_1 \quad (24)$$

so that  $\mathbf{v}^*$ , the velocity of  $P^*$ , is given by

$$\mathbf{v}^* = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \mathbf{t}_1 + \frac{m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)}{m_1 + m_2} \dot{\theta} \mathbf{t}_2 \quad (25)$$

(24) (4.4.5)

and the partial velocities of  $S^*$  are

$$\mathbf{v}_1^* = \frac{m_1}{m_1 + m_2} \mathbf{t}_1 \quad \mathbf{v}_2^* = \frac{m_2}{m_1 + m_2} \mathbf{t}_1 \quad (26)$$

Now one can refer to Eqs. (2) to write

$$(F_1)_{(2)} = [(m_1 + m_2)g\mathbf{k}] \cdot \left( \frac{m_1}{m_1 + m_2} \mathbf{t}_1 \right) = m_1 g \cos \theta \quad (27)$$

$$(F_2)_{(2)} = [(m_1 + m_2)g\mathbf{k}] \cdot \left( \frac{m_2}{m_1 + m_2} \mathbf{t}_1 \right) = m_2 g \cos \theta \quad (28)$$

These results agree with Eqs. (22) and (23), but more effort had to be expended to derive Eqs. (27) and (28) than to generate Eqs. (22) and (23).

## 4.9 BRINGING NONCONTRIBUTING FORCES INTO EVIDENCE

As was mentioned in Sec. 4.5, the fact that certain forces acting on the particles of a system make no contributions to generalized active forces usually is helpful. But it can occur that precisely such a noncontributing force, or a torque of a couple formed by noncontributing forces, is of interest in its own right. In that event, one can bring this force or torque into evidence through the introduction of a generalized speed properly related to the force or torque in question, that is, a generalized speed that gives rise to a partial velocity of the point of application of the force, or a partial angular velocity of the rigid body on which the couple acts, such that the dot-product of the partial velocity and the force, or the dot-product of the partial angular velocity and the torque, does not vanish. The force or torque of interest then comes into evidence in the generalized active force corresponding to the additional degree of freedom associated with the additional generalized speed.

The introduction of a suitable additional generalized speed is accomplished by permitting points to have certain velocities, or rigid bodies to have certain angular velocities, which they cannot, in fact possess, doing so *without* introducing additional generalized coordinates. When forming expressions for velocities and/or angular velocities, one then takes the additional generalized speed into account, but one uses the same generalized coordinates as before, and one identifies partial velocities and partial angular velocities corresponding to the additional generalized speed by inspection, as always. The partial velocities and partial angular velocities corresponding to the original generalized speeds, as well as the associated original generalized active forces, remain unaltered.

**Example** In Fig. 4.9.1,  $P$  is a particle of mass  $m$  that can slide freely on a smooth, uniform rod  $R$  of mass  $M$  and length  $2L$ .  $R$  is rigidly attached at an angle  $\beta$  to a light sleeve  $S$  that is supported by a smooth, fixed vertical shaft  $V$  and a smooth bearing surface  $B$ , and  $S$  is subjected to the action of a couple whose torque  $\mathbf{T}$  is given by

$$\mathbf{T} = T\mathbf{s}_1 \quad (1)$$

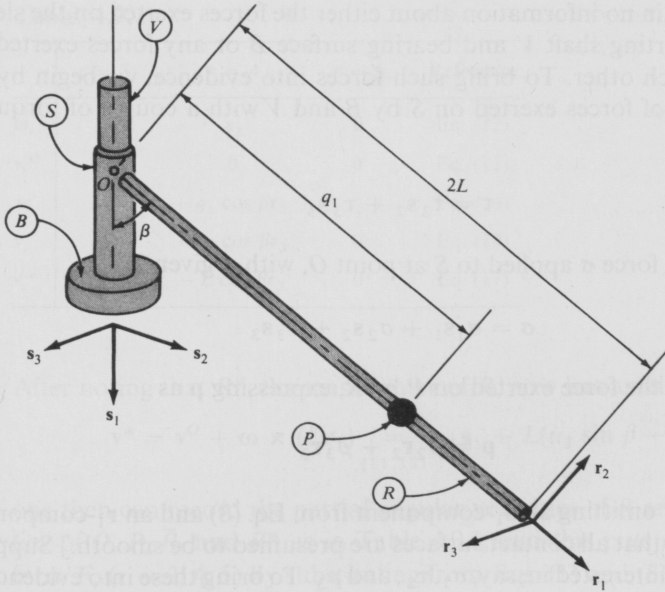


Figure 4.9.1

where  $T$  is time-dependent and  $\mathbf{s}_1$  is a unit vector directed vertically downward. The unit vectors  $\mathbf{s}_2$ ,  $\mathbf{s}_3$ ,  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $\mathbf{r}_3$  in Fig. 4.9.1 are defined as follows:  $\mathbf{s}_2$  is perpendicular to  $\mathbf{s}_1$  and parallel to the plane determined by the axes of  $S$  and  $R$ ;  $\mathbf{s}_3 = \mathbf{s}_1 \times \mathbf{s}_2$ ;  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $\mathbf{r}_3$  form a dextral set of mutually perpendicular unit vectors, with  $\mathbf{r}_1$  parallel to the axis of  $R$  and  $\mathbf{r}_2$  parallel to the plane determined by the axes of  $S$  and  $R$ .

The system formed by  $P$ ,  $R$ , and  $S$  possesses two degrees of freedom and, if generalized speeds  $u_1$  and  $u_2$  are defined as

$$u_1 \triangleq \dot{q}_1 \quad u_2 \triangleq \boldsymbol{\omega} \cdot \mathbf{s}_1 \quad (2)$$

where  $q_1$  is the distance from  $O$  to  $P$  and  $\boldsymbol{\omega}$  is the angular velocity of  $R$ , then the associated partial angular velocities of  $R$  and partial velocities of  $P$  are

$$\boldsymbol{\omega}_1 = 0 \quad \mathbf{v}_1 = \mathbf{r}_1 \quad (3)$$

$$\boldsymbol{\omega}_2 = \mathbf{s}_1 \quad \mathbf{v}_2 = q_1 \sin \beta \mathbf{r}_3 \quad (4)$$

Hence, the generalized active forces  $F_1$  and  $F_2$ , found by substituting from Eqs. (1), (3), and (4) into [see Eqs. (4.6.1) and (4.8.1)]

$$F_r = \boldsymbol{\omega}_r \cdot \mathbf{T} + \mathbf{v}_r \cdot (mg\mathbf{s}_1) \quad (r = 1, 2) \quad (5)$$

are

$$F_1 = mg \cos \beta \quad (6)$$

$$F_2 = T \quad (7)$$

and these contain no information about either the forces exerted on the sleeve  $S$  by the supporting shaft  $V$  and bearing surface  $B$  or any forces exerted by  $P$  and  $R$  on each other. To bring such forces into evidence, we begin by replacing the set of forces exerted on  $S$  by  $B$  and  $V$  with a couple of torque  $\tau$ , expressed as

$$\tau = \tau_2 \mathbf{s}_2 + \tau_3 \mathbf{s}_3 \quad (8)$$

together with a force  $\sigma$  applied to  $S$  at point  $O$ , with  $\sigma$  given by

$$\sigma = \sigma_1 \mathbf{s}_1 + \sigma_2 \mathbf{s}_2 + \sigma_3 \mathbf{s}_3 \quad (9)$$

and we let  $\rho$  be the force exerted on  $P$  by  $R$ , expressing  $\rho$  as

$$\rho = \rho_2 \mathbf{r}_2 + \rho_3 \mathbf{r}_3 \quad (10)$$

[The reason for omitting an  $\mathbf{s}_1$ -component from Eq. (8) and an  $\mathbf{r}_1$ -component from Eq. (10) is that all contact surfaces are presumed to be smooth.] Suppose now that one is interested in, say,  $\sigma_1$ ,  $\tau_2$ , and  $\rho_3$ . To bring these into evidence in expressions for generalized forces, one permits  $S$  to move in such a way that the point  $O$ , regarded as a point of  $R$ , has a velocity  $\mathbf{v}^O$  given by

$$\mathbf{v}^O = u_3 \mathbf{s}_1 \quad (11)$$

while  $R$  has an angular velocity  $\omega$  that can be expressed as

$$\omega = u_2 \mathbf{s}_1 + u_4 \mathbf{s}_2 \quad (12)$$

Also, one allows  $P$  to have a velocity  $\mathbf{v}$  such that  $P$  can lose contact with  $R$  or can penetrate  $R$  by moving in the direction of  $\mathbf{r}_3$ , which is accomplished by letting  $\bar{\mathbf{v}}$  denote the velocity of the point  $\bar{R}$  of  $R$  with which  $P$  is in contact and then writing

$$\mathbf{v} = \bar{\mathbf{v}} + {}^R\mathbf{v}^P \quad (13)$$

(2.8.1)

with

$$\bar{\mathbf{v}} = \mathbf{v}^O + \omega \times (q_1 \mathbf{r}_1) = u_3 \mathbf{s}_1 + q_1(u_2 \sin \beta - u_4 \cos \beta) \mathbf{r}_3 \quad (14)$$

(2.7.1) (11,12)

and

$${}^R\mathbf{v}^P = \dot{q}_1 \mathbf{r}_1 + u_5 \mathbf{r}_3 = u_1 \mathbf{r}_1 + u_5 \mathbf{r}_3 \quad (15)$$

(2)

so that

$$\mathbf{v} = u_1 \mathbf{r}_1 + u_2 q_1 \sin \beta \mathbf{r}_3 + u_3 \mathbf{s}_1 - u_4 q_1 \cos \beta \mathbf{r}_3 + u_5 \mathbf{r}_3 \quad (16)$$

(13-15)

**Table 4.9.1**

	$r = 3$	$r = 4$	$r = 5$	Reference
$\omega_r$	0	$s_2$	0	Eq. (12)
$v_r^O$	$s_1$	0	0	Eq. (11)
$v_r$	$s_1$	$-q_1 \cos \beta \mathbf{r}_3$	$\mathbf{r}_3$	Eq. (16)
$\bar{v}_r$	$s_1$	$-q_1 \cos \beta \mathbf{r}_3$	0	Eq. (14)
$v_r^*$	$s_1$	$-L \cos \beta \mathbf{r}_3$	0	Eq. (17)

After noting that  $R^*$ , the mass center of  $R$ , now has a velocity  $\mathbf{v}^*$  given by

$$\mathbf{v}^* = \mathbf{v}^O + \boldsymbol{\omega} \times (L\mathbf{r}_1) = u_3 \mathbf{s}_1 + L(u_2 \sin \beta - u_4 \cos \beta) \mathbf{r}_3 \quad (17)$$

(11,12)

one then can record the partial angular velocities of  $R$  and the partial velocities of  $O$ ,  $P$ ,  $\bar{R}$ , and  $R^*$  as in Table 4.9.1, and this puts one into position to form  $F_r$  ( $r = 3, 4, 5$ ) by substituting from Eqs. (1) and (8)–(10) into

$$F_r = \boldsymbol{\omega}_r \cdot (\mathbf{T} + \boldsymbol{\tau}) + \mathbf{v}_r^O \cdot \boldsymbol{\sigma} + \mathbf{v}_r \cdot (mg\mathbf{s}_1 + \boldsymbol{\rho}) + \bar{\mathbf{v}}_r \cdot (-\boldsymbol{\rho}) + \mathbf{v}_r^* \cdot (Mg\mathbf{s}_1) \quad (r = 3, 4, 5) \quad (18)$$

which leads to

$$F_3 = \sigma_1 + (m + M)g \quad (19)$$

$$F_4 = \tau_2 \quad (20)$$

$$F_5 = \rho_3 \quad (21)$$

The quantities  $\sigma_2$ ,  $\sigma_3$ ,  $\tau_3$ , and  $\rho_2$ , which are absent from  $F_1, \dots, F_5$ , can be brought into evidence similarly; that is, if  $u_6, \dots, u_9$  are introduced such that

$$\mathbf{v}^O = u_3 \mathbf{s}_1 + u_6 \mathbf{s}_2 + u_7 \mathbf{s}_3 \quad (22)$$

$$\boldsymbol{\omega} = u_2 \mathbf{s}_1 + u_4 \mathbf{s}_2 + u_8 \mathbf{s}_3 \quad (23)$$

and

$${}^R\mathbf{v}^P = u_1 \mathbf{r}_1 + u_9 \mathbf{r}_2 + u_5 \mathbf{r}_3 \quad (24)$$

then the generalized active forces corresponding to  $u_6, \dots, u_9$  are

$$F_6 = \sigma_2 \quad (25)$$

$$F_7 = \sigma_3 \quad (26)$$

$$F_8 = \tau_3 - (mq_1 + ML)g \sin \beta \quad (27)$$

$$F_9 = \rho_2 - mg \sin \beta \quad (28)$$

## 4.10 COULOMB FRICTION FORCES

Suppose that a particle  $P$  or a rigid body  $B$  belonging to a simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a reference frame  $A$  (see Sec. 2.13) is in contact with a rigid body  $C$  (which may, or may not, belong to  $S$ ). Then, if  $P$  or  $B$  is sliding on  $C$ , the contributions of contact forces exerted on  $P$  or  $B$  by  $C$  to the generalized active forces  $F_1, \dots, F_p$  (see Sec. 4.4) depend on both the magnitudes and the directions of such contact forces. When contact takes place across dry, clean surfaces, certain information regarding the magnitudes and the directions of contact forces can be obtained from the *laws of Coulomb friction*, which will now be stated.

When a particle  $P$  that is in contact with a rigid body  $C$  is at rest relative to  $C$ , then  $C$  exerts on  $P$  a contact force  $\mathbf{C}$  that can be expressed as

$$\mathbf{C} = N\mathbf{v} + T\boldsymbol{\tau} \quad (1)$$

where  $\mathbf{v}$  is a unit vector normal to the surface  $\Sigma$  of  $C$  at  $P$  and directed from  $C$  toward  $P$ ,  $\boldsymbol{\tau}$  is a unit vector perpendicular to  $\mathbf{v}$ ,  $N$  is non-negative, and  $T$  satisfies the inequality

$$|T| \leq \mu N \quad (2)$$

in which  $\mu$ , called the *coefficient of static friction* for  $P$  and  $C$ , is a quantity whose value depends solely on the materials of which  $P$  and  $C$  are made. Typical values are 0.2 for metal on metal, 0.6 for metal on wood.

When  $P$  is in a state of impending tangential motion relative to  $C$ , that is, when  $P$  is on the verge of moving tangentially relative to  $C$ , then

$$|T| = \mu N \quad (3)$$

and the vector  $T\boldsymbol{\tau}$  appearing in Eq. (1) points in the direction opposite to that in which  $P$  is about to move relative to  $C$ .

When  $P$  is sliding relative to  $C$ , Eq. (1) remains in force, but the inequality (2) gives way to the equality

$$|T| = \mu' N \quad (4)$$

where  $\mu'$ , called the *coefficient of kinetic friction* for  $P$  and  $C$ , has a value generally smaller than that of  $\mu$ ; and the vector  $T\boldsymbol{\tau}$  in Eq. (1) now is directed oppositely to  ${}^C\mathbf{v}^P$ , the velocity of  $P$  in  $C$ .

When a rigid body  $B$ , rather than a particle  $P$ , is in contact with  $C$ , the same laws apply if the surface  $\Sigma$  over which  $B$  and  $C$  are in contact has an area so small that  $\Sigma$  can be regarded as a point. Otherwise, that is, when  $\Sigma$  has an area that cannot be regarded as negligibly small, the laws already stated apply in connection with every differential element of  $\Sigma$ . More specifically, if  $P$  is a point of  $B$  within a portion  $\bar{\Sigma}$  of  $\Sigma$  that has an area  $\bar{A}$ , then the set of contact forces exerted on  $B$  by  $C$  across  $\bar{\Sigma}$  can be replaced with a couple of torque  $\mathbf{M}$  together with a force  $\mathbf{C}$

whose line of action passes through  $P$ ;  $\mathbf{M}$  and  $\mathbf{C}$  depend on  $\bar{A}$ , and both approach zero as  $\bar{A}$  approaches zero, but

$$\lim_{\bar{A} \rightarrow 0} \frac{\mathbf{M}}{\bar{A}} = 0 \quad (5)$$

whereas  $\mathbf{C}/\bar{A}$  has a nonzero limit that can be expressed as

$$\lim_{\bar{A} \rightarrow 0} \frac{\mathbf{C}}{\bar{A}} = n\mathbf{v} + t\boldsymbol{\tau} \quad (6)$$

where  $n$ , called the *pressure* at point  $P$ , and  $t$ , called the *shear* at  $P$ , depend on the position of  $P$  within  $\Sigma$ ,  $n$  is non-negative, and  $\mathbf{v}$  and  $\boldsymbol{\tau}$  have the same meanings as before. Equations (5) and (6) imply that, if  $P$  is a point of  $B$  lying within a differential element  $d\Sigma$  of  $\Sigma$  having an area  $dA$ , the set of contact forces exerted on  $B$  by  $C$  across  $d\Sigma$  is equivalent to a force  $d\mathbf{C}$  whose line of action passes through  $P$  and that is given by

$$d\mathbf{C} = (n\mathbf{v} + t\boldsymbol{\tau}) dA \quad (7)$$

This equation takes the place of Eq. (1) when  $B$  and  $C$  are in contact over an extended surface, and the relationships (2)–(4) then are replaced with, respectively,

$$|t| \leq \mu n \quad (8)$$

when  $P$  is at rest relative to  $C$ ,

$$|t| = \mu n \quad (9)$$

when  $P$  is in a state of impending tangential motion relative to  $C$ , and

$$|t| = \mu' n \quad (10)$$

when  $P$  is sliding relative to  $C$ .

When a particle  $P$  is in contact with a rigid body  $C$  modeled as matter distributed along a curve  $\Gamma$  (for example, a thin rod modeled as matter distributed along a straight line), then Eqs. (1)–(4) and the statements made in connection with these equations remain in force, provided that  $\boldsymbol{\tau}$  is regarded as a unit vector tangent to  $\Gamma$  at  $P$  while  $\mathbf{v}$  is simply a unit vector perpendicular to  $\boldsymbol{\tau}$ . Similarly, when contact between two rigid bodies  $B$  and  $C$  is regarded as taking place along a curve  $\Gamma$  (for example, when a generator of a right-circular cylinder is in contact with a plane), then Eq. (7) applies after  $dA$  has been replaced with  $dL$ , the length of a differential element  $d\Gamma$  of  $\Gamma$ , and  $\mathbf{v}$  and  $\boldsymbol{\tau}$  then have their original meanings.

**Example** Considering once again the system formed by the sleeve  $S$ , rod  $R$ , and particle  $P$  depicted in Fig. 4.9.1 and considered previously in the example in Sec. 4.9, suppose that the contact between  $S$  and the bearing surface  $B$ , as well as that between  $P$  and  $R$ , takes place across a rough, rather than a smooth, surface, but that the vertical shaft  $V$  can be regarded as smooth, as

heretofore. Then, when  $S$  is moving relative to  $B$ , and  $P$  relative to  $R$ , contact forces that contribute to the generalized active forces  $F_1$  and  $F_2$  come into play.  $(F_r)_C$ , the contribution to  $F_r$  ( $r = 1, 2$ ) of the contact forces, will now be determined.

The contact force  $\boldsymbol{\rho}$  exerted by  $R$  on  $P$  can be expressed as

$$\boldsymbol{\rho} = \rho_1 \mathbf{r}_1 + \rho_2 \mathbf{r}_2 + \rho_3 \mathbf{r}_3 \quad (11)$$

and the contact force  $\bar{\boldsymbol{\rho}}$  exerted by  $P$  on  $R$  is given by

$$\bar{\boldsymbol{\rho}} = -\boldsymbol{\rho} \quad (12)$$

Hence, letting  $\bar{\mathbf{v}}_r$  ( $r = 1, 2$ ) denote the partial velocities of the point of  $R$  at which  $\bar{\boldsymbol{\rho}}$  is applied to  $R$ , so that

$$\bar{\mathbf{v}}_1 = 0 \quad \bar{\mathbf{v}}_2 = q_1 \sin \beta \mathbf{r}_3 \quad (13)$$

one can express the contributions of  $\boldsymbol{\rho}$  and  $\bar{\boldsymbol{\rho}}$  to the generalized active forces  $F_1$  as

$$\mathbf{v}_1 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_1 \cdot \bar{\boldsymbol{\rho}} = \underbrace{\mathbf{r}_1 \cdot \boldsymbol{\rho}}_{(4.9.3)} + \underbrace{0}_{(13)(11)} = \rho_1 \quad (14)$$

and

$$\mathbf{v}_2 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_2 \cdot \bar{\boldsymbol{\rho}} = \underbrace{(\mathbf{v}_2 - \bar{\mathbf{v}}_2) \cdot \boldsymbol{\rho}}_{(12)} = \underbrace{0}_{(4.9.4,13)} \quad (15)$$

The laws of friction make it possible to express  $\rho_1$  in terms of  $\rho_2$  and  $\rho_3$ , for  $\rho_1 \mathbf{r}_1$  in Eq. (11) corresponds to the second term of Eq. (1) while  $\rho_2 \mathbf{r}_2 + \rho_3 \mathbf{r}_3$  plays the part of the first term. Accordingly,

$$|\rho_1| = \underbrace{\mu_1'}_{(4)} (\rho_2^2 + \rho_3^2)^{1/2}_{(11)} \quad (16)$$

where  $\mu_1'$  is the coefficient of kinetic friction for  $P$  and  $R$ . Furthermore,  $\rho_1 \mathbf{r}_1$  must have a direction opposite to that of  ${}^R \mathbf{v}^P$ , the velocity of  $P$  in  $R$ , which means that  $\rho_1 \mathbf{r}_1$  can be written

$$\begin{aligned} \rho_1 \mathbf{r}_1 &= -|\rho_1| \frac{{}^R \mathbf{v}^P}{|{}^R \mathbf{v}^P|}_{(16)} = -\mu_1' (\rho_2^2 + \rho_3^2)^{1/2} \frac{u_1 \mathbf{r}_1}{|u_1|}_{(4.9.2)} \\ &= -\mu_1' (\rho_2^2 + \rho_3^2)^{1/2} \operatorname{sgn} u_1 \mathbf{r}_1 \end{aligned} \quad (17)$$

from which it follows that

$$\rho_1 = -\mu_1' (\rho_2^2 + \rho_3^2)^{1/2} \operatorname{sgn} u_1 \quad (18)$$

so that

$$\mathbf{v}_1 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_1 \cdot \bar{\boldsymbol{\rho}} = \underbrace{-\mu_1' (\rho_2^2 + \rho_3^2)^{1/2} \operatorname{sgn} u_1}_{(14,18)} \quad (19)$$

To deal with the contributions to  $F_1$  and  $F_2$  of the contact forces exerted on  $S$  by  $B$ , we let  $b_1$  and  $b_2$  be the inner and outer radii of  $S$ , respectively, and



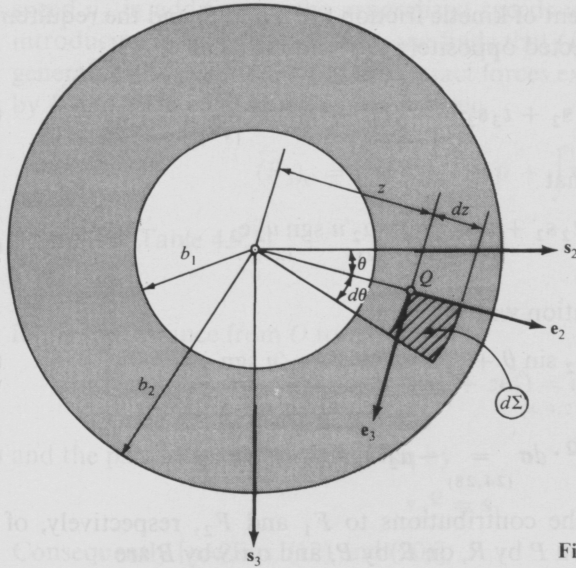


Figure 4.10.1

consider a generic differential element  $d\Sigma$  of the surface  $\Sigma$  of  $S$  that is in contact with  $B$ ,  $d\Sigma$  having the dimensions indicated in Fig. 4.10.1, where  $\theta$  and  $z$  are variables used to locate one corner,  $Q$ , of  $d\Sigma$ . The force  $d\sigma$  exerted on  $S$  by  $B$  across  $d\Sigma$  then can be written

$$d\sigma = (-ns_1 + t_2s_2 + t_3s_3) z dz d\theta \quad (20)$$

(7)

where  $-s_1$ , which points from  $B$  toward  $S$ , corresponds to  $\mathbf{v}$  in Eq. (7) since  $Q$  is a point of  $d\Sigma$ , and  $\mathbf{v}^Q$ , the velocity of  $Q$ , is given by (see Fig. 4.10.1 for the unit vectors  $\mathbf{e}_2$  and  $\mathbf{e}_3$ )

$$\mathbf{v}^Q = \boldsymbol{\omega} \times (z\mathbf{e}_2) = (u_2\mathbf{s}_1) \times (z\mathbf{e}_2) = u_2 z \mathbf{e}_3 \quad (21)$$

(4.9.2)

so that the partial velocities of  $Q$  are

$$\mathbf{v}_1^Q = 0 \quad \mathbf{v}_2^Q = z\mathbf{e}_3 \quad (22)$$

We can now express the contributions of  $d\sigma$  to  $F_1$  and  $F_2$  as

$$\mathbf{v}_1^Q \cdot d\sigma = 0 \quad (23)$$

(22)

and

$$\mathbf{v}_2^Q \cdot d\sigma = z^2(-t_2 \sin \theta + t_3 \cos \theta) dz d\theta \quad (24)$$

(22,20)

In addition, we have the relationship

$$(t_2^2 + t_3^2)^{1/2} = \mu_2' n \quad (25)$$

(10)

where  $\mu_2'$  is the coefficient of kinetic friction for  $B$  and  $S$ , and the requirement that  $t_2 \mathbf{s}_2 + t_3 \mathbf{s}_3$  be directed oppositely to  $\mathbf{v}^Q$  implies that

$$t_2 \mathbf{s}_2 + t_3 \mathbf{s}_3 = -(t_2^2 + t_3^2)^{1/2} \frac{\mathbf{v}^Q}{|\mathbf{v}^Q|} \quad (26)$$

from which it follows that

$$t_2 \mathbf{s}_2 + t_3 \mathbf{s}_3 = \underset{(26)}{-\mu_2' n} \underset{(25)}{\text{sgn } u_2} \underset{(21)}{\mathbf{e}_3} \quad (27)$$

or, upon dot-multiplication with  $\mathbf{e}_3$ , that

$$-t_2 \sin \theta + t_3 \cos \theta = -\mu_2' n \text{sgn } u_2 \quad (28)$$

Hence,

$$\mathbf{v}_2^Q \cdot d\boldsymbol{\sigma} = \underset{(24,28)}{-\mu_2' \text{sgn } u_2 n z^2 dz d\theta} \quad (29)$$

$(F_1)_C$  and  $(F_2)_C$ , the contributions to  $F_1$  and  $F_2$ , respectively, of the contact forces exerted on  $P$  by  $R$ , on  $R$  by  $P$ , and on  $S$  by  $B$  are

$$\begin{aligned} (F_1)_C &= \mathbf{v}_1 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_1 \cdot \bar{\boldsymbol{\rho}} + \int \mathbf{v}_1^Q \cdot d\boldsymbol{\sigma} \\ &= \underset{(19,23)}{-\mu_1'(\rho_2^2 + \rho_3^2)^{1/2} \text{sgn } u_1} \end{aligned} \quad (30)$$

and

$$\begin{aligned} (F_2)_C &= \mathbf{v}_2 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_2 \cdot \bar{\boldsymbol{\rho}} + \int \mathbf{v}_2^Q \cdot d\boldsymbol{\sigma} \\ &= \underset{(15,29)}{-\mu_2' \text{sgn } u_2} \int_0^{2\pi} \int_{b_1}^{b_2} n z^2 dz d\theta \end{aligned} \quad (31)$$

The definite integral in Eq. (31) can be evaluated only when the pressure  $n$  is known as a function of  $z$  and  $\theta$ . To discover this functional relationship, one must use the methods of the theory of elasticity. However, satisfactory results often can be obtained in this sort of situation by making a relatively simple assumption regarding the pressure distribution in question, such as, for example, that  $n$  is independent of  $z$  and  $\theta$  and thus has a value  $n^*$  that depends solely on the time  $t$ . Under these circumstances,

$$\begin{aligned} (F_2)_C &= \underset{(31)}{-\mu_2' \text{sgn } u_2 n^*} \int_0^{2\pi} \int_{b_1}^{b_2} z^2 dz d\theta \\ &= -\frac{2\pi n^*}{3} (b_2^3 - b_1^3) \mu_2' \text{sgn } u_2 \end{aligned} \quad (32)$$

As will be seen later,  $n^*$  ultimately can be determined by resorting to the procedure employed in Sec. 4.9, that is, by introducing a suitable generalized

speed  $u_3$  in addition to the generalized speeds  $u_1$  and  $u_2$ . Specifically, after introducing  $u_3$  as in Eq. (4.9.11), one finds that  $(F_3)_C$ , the contribution to the generalized active force  $F_3$  of all contact forces exerted by  $B$  and  $V$  on  $S$ , and by  $R$  and  $P$  on each other, can be written

$$(F_3)_C = \mathbf{v}_3 \cdot \boldsymbol{\rho} + \bar{\mathbf{v}}_3 \cdot \bar{\boldsymbol{\rho}} + \int \mathbf{v}_3^Q \cdot d\boldsymbol{\sigma} \quad (33)$$

where (see Table 4.9.1)

$$\mathbf{v}_3 = \bar{\mathbf{v}}_3 \quad (34)$$

If  $h$  is the distance from  $O$  to  $\Sigma$ , then

$$\mathbf{v}_3^Q = \mathbf{v}^O + \boldsymbol{\omega} \times (h\mathbf{s}_1 + z\mathbf{e}_2) = u_3\mathbf{s}_1 + u_2z\mathbf{e}_3 \quad (35)$$

(2.7.1) (4.9.2)

and the partial velocity  $\mathbf{v}_3^Q$  is given by

$$\mathbf{v}_3^Q = \mathbf{s}_1 \quad (36)$$

Consequently [see Eqs. (12) and (20)],

$$\begin{aligned} (F_3)_C &= -n^* \int_0^{2\pi} \int_{b_1}^{b_2} z \, dz \, d\theta \\ &= -\pi n^* (b_2^2 - b_1^2) \end{aligned} \quad (37)$$

(33, 34, 36)

The quantities  $\rho_2$  and  $\rho_3$  appearing in Eq. (30) come into evidence also when one forms the contributions  $(F_5)_C$  and  $(F_9)_C$  of contact forces to the generalized active forces  $F_5$  and  $F_9$  corresponding to  $u_5$  and  $u_9$ , respectively, where  $u_5$  and  $u_9$  have the same meaning as in Eq. (4.9.24). Specifically, in agreement with Eqs. (4.9.21) and (4.9.28),

$$(F_5)_C = \rho_3 \quad (F_9)_C = \rho_2 \quad (38)$$

Expressions for the contributions to  $F_r$  ( $r = 1, 2, 3, 5, 9$ ) of all gravitational forces acting on  $S$ ,  $R$ , and  $P$  and of the torque  $\mathbf{T}$  described by Eq. (4.9.1) can be formulated as they were in connection with Eqs. (4.9.6), (4.9.7), (4.9.19), (4.9.21), and (4.9.28). Hence, the complete generalized active forces are

$$F_1 = mg \cos \beta - \mu_1'(\rho_2^2 + \rho_3^2)^{1/2} \operatorname{sgn} u_1 \quad (39)$$

(4.9.6) (30)

$$F_2 = T - \frac{2\pi n^*}{3} (b_2^3 - b_1^3) \mu_2' \operatorname{sgn} u_2 \quad (40)$$

(4.9.7) (32)

$$F_3 = (m + M)g - \pi n^* (b_2^2 - b_1^2) \quad (41)$$

(4.9.19) (37)

$$F_5 = \rho_3 \quad (42)$$

(4.9.21, 38)

$$F_9 = -mg \sin \beta + \rho_2 \quad (43)$$

(4.9.28) (38)

These results will be used in Sec. 6.3 in the formulation of equations of motion of the system formed by  $S$ ,  $R$ , and  $P$ .

#### 4.11 GENERALIZED INERTIA FORCES

If  $u_1, \dots, u_n$  are generalized speeds for a simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a reference frame  $A$  (see Sec. 2.13),  $p$  quantities  $\tilde{F}_1^*, \dots, \tilde{F}_p^*$ , called *nonholonomic generalized inertia forces* for  $S$  in  $A$ , and  $n$  quantities  $F_1^*, \dots, F_n^*$ , called *holonomic generalized inertia forces* for  $S$  in  $A$ , are defined as

$$\tilde{F}_r^* \triangleq \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i^* \quad (r = 1, \dots, p) \quad (1)$$

and

$$F_r^* = \sum_{i=1}^v \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i^* \quad (r = 1, \dots, n) \quad (2)$$

respectively, where  $v$  is the number of particles comprising  $S$ ,  $P_i$  is a typical particle of  $S$ ,  $\tilde{\mathbf{v}}_r^{P_i}$  and  $\mathbf{v}_r^{P_i}$  are, respectively, a nonholonomic partial velocity of  $P_i$  in  $A$  and a holonomic partial velocity of  $P_i$  in  $A$  (see Sec. 2.14), and  $\mathbf{R}_i^*$  is the *inertia force* for  $P_i$  in  $A$ ; that is,

$$\mathbf{R}_i^* \triangleq -m_i \mathbf{a}_i \quad (i = 1, \dots, v) \quad (3)$$

where  $m_i$  is the mass of  $P_i$ , and  $\mathbf{a}_i$  is the acceleration of  $P_i$  in  $A$ .

Nonholonomic and holonomic generalized inertia forces for  $S$  in  $A$  are related to each other and to the quantities  $A_{rs}$  ( $s = 1, \dots, p$ ;  $r = p + 1, \dots, n$ ) introduced in Eqs. (2.13.1), as follows:

$$\tilde{F}_r^* = F_r^* + \sum_{s=p+1}^n F_s^* A_{sr} \quad (r = 1, \dots, p) \quad (4)$$

$(\tilde{F}_r^*)_B$ , the contribution to  $\tilde{F}_r^*$  ( $r = 1, \dots, p$ ) of all inertia forces for the particles of a rigid body  $B$  belonging to  $S$ , can be expressed in terms of  $\mathbf{T}^*$  and  $\mathbf{R}^*$  defined, respectively, as

$$\mathbf{T}^* \triangleq - \sum_{i=1}^{\beta} m_i \mathbf{r}_i \times \mathbf{a}_i \quad (5)$$

and

$$\mathbf{R}^* \triangleq -M \mathbf{a}^* \quad (6)$$

where  $\beta$  is the number of particles forming  $B$ ,  $m_i$  is the mass of a generic particle  $P_i$  of  $B$ ,  $\mathbf{r}_i$  is the position vector from  $B^*$ , the mass center of  $B$ , to  $P_i$ ,  $\mathbf{a}_i$  is the acceleration of  $P_i$  in  $A$ ,  $M$  is the total mass of  $B$ , and  $\mathbf{a}^*$  is the acceleration of  $B^*$  in  $A$ .

$\mathbf{T}^*$  and  $\mathbf{R}^*$  are called, respectively, the *inertia torque* for  $B$  in  $A$  and the *inertia force* for  $B$  in  $A$ , and  $(\tilde{F}_r^*)_B$  can be written

$$(\tilde{F}_r^*)_B = \tilde{\omega}_r \cdot \mathbf{T}^* + \tilde{\mathbf{v}}_r^* \cdot \mathbf{R}^* \quad (r = 1, \dots, p) \quad (7)$$

where  $\tilde{\omega}_r$  and  $\tilde{\mathbf{v}}_r^*$  are, respectively, the  $r$ th nonholonomic partial angular velocity of  $B$  in  $A$  and the  $r$ th nonholonomic partial velocity of  $B^*$  in  $A$  (see Sec. 2.14).

To use Eqs. (7) effectively, one must take advantage of the fact that  $\mathbf{T}^*$  can be expressed in a number of ways making it unnecessary to perform explicitly the summation indicated in Eq. (5). For example,

$$\mathbf{T}^* = -\boldsymbol{\alpha} \cdot \mathbf{I} - \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} \quad (8)$$

where  $\boldsymbol{\alpha}$  and  $\boldsymbol{\omega}$  are, respectively, the angular acceleration of  $B$  in  $A$  and the angular velocity of  $B$  in  $A$ , and  $\mathbf{I}$  is the central inertia dyadic of  $B$  (see Sec. 3.5). If  $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$  form a dextral set of mutually perpendicular unit vectors, each parallel to a central principal axis of  $B$  (see Sec. 3.8), but not necessarily fixed in  $B$ , and  $\alpha_j, \omega_j$ , and  $I_j$  are defined as

$$\alpha_j \triangleq \boldsymbol{\alpha} \cdot \mathbf{c}_j \quad \omega_j \triangleq \boldsymbol{\omega} \cdot \mathbf{c}_j \quad I_j \triangleq \mathbf{c}_j \cdot \mathbf{I} \cdot \mathbf{c}_j \quad (j = 1, 2, 3) \quad (9)$$

then Eq. (8) can be replaced with

$$\begin{aligned} \mathbf{T}^* = & -[\alpha_1 I_1 - \omega_2 \omega_3 (I_2 - I_3)] \mathbf{c}_1 \\ & -[\alpha_2 I_2 - \omega_3 \omega_1 (I_3 - I_1)] \mathbf{c}_2 \\ & -[\alpha_3 I_3 - \omega_1 \omega_2 (I_1 - I_2)] \mathbf{c}_3 \end{aligned} \quad (10)$$

When generalized speeds have been introduced in addition to  $u_1, \dots, u_p$  for the purpose of bringing into evidence forces and/or torques that contribute nothing to the generalized active forces  $\tilde{F}_1, \dots, \tilde{F}_p$ , or, if  $S$  is a holonomic system, to  $F_1, \dots, F_n$  (see Sec. 4.9), then the generalized inertia forces corresponding to the additional generalized speeds are found by using in Eqs. (1), (2), and (7) partial velocities and partial angular velocities formed as explained in Sec. 4.9. The expressions to be used here for the vectors  $\mathbf{a}_i, \mathbf{a}^*, \boldsymbol{\alpha}$ , and  $\boldsymbol{\omega}$  appearing variously in Eqs. (3), (5), (6), and (8) are precisely those employed in forming the generalized inertia forces  $\tilde{F}_1^*, \dots, \tilde{F}_p^*$ , or, if  $S$  is a holonomic system,  $F_1^*, \dots, F_n^*$ .

**Derivations** To establish the validity of Eqs. (4), one can proceed as in the derivation of Eqs. (4.4.3).

With  $\beta, m_i$ , and  $\mathbf{r}_i$  as defined,

$$\sum_{i=1}^{\beta} m_i \mathbf{r}_i = 0 \quad (3.1) \quad (11)$$

and  $\mathbf{a}_i$  can be expressed as

$$\mathbf{a}_i = \mathbf{a}^* + \boldsymbol{\alpha} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) \quad (i = 1, \dots, \beta) \quad (12)$$

(2.7.2)

Consequently,

$$\begin{aligned} \sum_{i=1}^{\beta} m_i \mathbf{a}_i &= \sum_{(12)}^{\beta} m_i \mathbf{a}^* + \boldsymbol{\alpha} \times \sum_{i=1}^{\beta} m_i \mathbf{r}_i + \boldsymbol{\omega} \times \left( \boldsymbol{\omega} \times \sum_{i=1}^{\beta} m_i \mathbf{r}_i \right) \\ &= \underset{(11)}{M} \mathbf{a}^* = \underset{(6)}{-\mathbf{R}^*} \end{aligned} \quad (13)$$

Now,

$$(\tilde{F}_r^*)_B = - \sum_{(1,3)}^{\beta} m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{a}_i \quad (r = 1, \dots, p) \quad (14)$$

and, since the velocity  $\mathbf{v}^{P_i}$  of  $P_i$  in  $A$  and the velocity  $\mathbf{v}^*$  of  $B^*$  in  $A$  are related by

$$\mathbf{v}^{P_i} = \mathbf{v}^* + \boldsymbol{\omega} \times \mathbf{r}_i \quad (2.7.1) \quad (15)$$

it follows from Eqs. (2.14.4) and (2.14.3) that

$$\tilde{\mathbf{v}}_r^{P_i} = \tilde{\mathbf{v}}_r^* + \tilde{\boldsymbol{\omega}}_r \times \mathbf{r}_i \quad (i = 1, \dots, \beta; r = 1, \dots, p) \quad (16)$$

Hence,

$$\begin{aligned} (\tilde{F}_r^*)_B &= - \sum_{(14)}^{\beta} m_i (\tilde{\mathbf{v}}_r^* + \tilde{\boldsymbol{\omega}}_r \times \mathbf{r}_i) \cdot \mathbf{a}_i \\ &= -\tilde{\mathbf{v}}_r^* \cdot \sum_{i=1}^{\beta} m_i \mathbf{a}_i - \tilde{\boldsymbol{\omega}}_r \cdot \sum_{i=1}^{\beta} m_i \mathbf{r}_i \times \mathbf{a}_i \\ &= \underset{(13)}{\tilde{\mathbf{v}}_r^* \cdot \mathbf{R}^*} + \underset{(5)}{\tilde{\boldsymbol{\omega}}_r \cdot \mathbf{T}^*} \quad (r = 1, \dots, p) \end{aligned} \quad (17)$$

which establishes the validity of Eqs. (7).

With the aid of Eqs. (12), one can express  $\mathbf{T}^*$  as

$$\begin{aligned} \mathbf{T}^* &= - \sum_{(5)}^{\beta} m_i \mathbf{r}_i \times [\mathbf{a}^* + \boldsymbol{\alpha} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= - \sum_{(11)}^{\beta} m_i \mathbf{r}_i \times (\boldsymbol{\alpha} \times \mathbf{r}_i) - \sum_{i=1}^{\beta} m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \end{aligned} \quad (18)$$

Now,

$$\begin{aligned} \sum_{i=1}^{\beta} m_i \mathbf{r}_i \times (\boldsymbol{\alpha} \times \mathbf{r}_i) &= \sum_{i=1}^{\beta} m_i (\mathbf{r}_i^2 \boldsymbol{\alpha} - \boldsymbol{\alpha} \cdot \mathbf{r}_i \mathbf{r}_i) \\ &= \boldsymbol{\alpha} \cdot \sum_{i=1}^{\beta} m_i (\mathbf{U} \mathbf{r}_i^2 - \mathbf{r}_i \mathbf{r}_i) = \underset{(3.5.16)}{\boldsymbol{\alpha} \cdot \mathbf{I}} \end{aligned} \quad (19)$$

and

$$\begin{aligned}
 \sum_{i=1}^{\beta} m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] &= - \sum_{i=1}^{\beta} m_i \mathbf{r}_i \cdot \boldsymbol{\omega} \boldsymbol{\omega} \times \mathbf{r}_i \\
 &= -\boldsymbol{\omega} \times \left[ \left( \sum_{i=1}^{\beta} m_i \mathbf{r}_i \mathbf{r}_i \right) \cdot \boldsymbol{\omega} \right] \\
 &= -\boldsymbol{\omega} \times \left\{ \left[ \sum_{i=1}^{\beta} m_i (\mathbf{r}_i \mathbf{r}_i - \mathbf{r}_i^2 \mathbf{U}) \right] \cdot \boldsymbol{\omega} \right\} \\
 &\stackrel{(3.5.16)}{=} \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega}
 \end{aligned} \tag{20}$$

Substitution from Eqs. (19) and (20) into Eq. (18) produces Eq. (8).

With  $\alpha_j$ ,  $\omega_j$ , and  $I_j$  as defined in Eqs. (9),

$$\boldsymbol{\alpha} = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2 + \alpha_3 \mathbf{c}_3 \tag{21}$$

$$\boldsymbol{\omega} = \omega_1 \mathbf{c}_1 + \omega_2 \mathbf{c}_2 + \omega_3 \mathbf{c}_3 \tag{22}$$

and

$$\mathbf{I} = I_1 \mathbf{c}_1 \mathbf{c}_1 + I_2 \mathbf{c}_2 \mathbf{c}_2 + I_3 \mathbf{c}_3 \mathbf{c}_3 \tag{23}$$

so that

$$\boldsymbol{\alpha} \cdot \mathbf{I} \stackrel{(21,23)}{=} \alpha_1 I_1 \mathbf{c}_1 + \alpha_2 I_2 \mathbf{c}_2 + \alpha_3 I_3 \mathbf{c}_3 \tag{24}$$

and

$$\boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} = -\omega_2 \omega_3 (I_2 - I_3) \mathbf{c}_1 - \omega_3 \omega_1 (I_3 - I_1) \mathbf{c}_2 - \omega_1 \omega_2 (I_1 - I_2) \mathbf{c}_3 \tag{25}$$

Equation (10) thus follows directly from Eqs. (8), (24), and (25).

**Example** For the system considered in the example in Sec. 4.9, the generalized inertia forces corresponding to  $u_1$  and  $u_2$  are given by

$$F_r^* \stackrel{(1,3,6,7)}{=} \mathbf{v}_r \cdot (-m\mathbf{a}) + \boldsymbol{\omega}_r \cdot \mathbf{T}^* + \mathbf{v}_r^* \cdot (-M\mathbf{a}^*) \quad (r = 1, 2) \tag{26}$$

where  $\mathbf{v}_r$  and  $\boldsymbol{\omega}_r$  ( $r = 1, 2$ ) are given in Eqs. (4.9.3) and (4.9.4),  $\mathbf{v}_r^*$  ( $r = 1, 2$ ) are partial velocities of the mass center of  $R$ ,  $\mathbf{a}$  is the acceleration of  $P$ ,  $\mathbf{a}^*$  is the acceleration of the mass center of  $R$ , and  $\mathbf{T}^*$  is the inertia torque for  $R$ . To construct the necessary expressions for  $\mathbf{v}_r^*$  ( $r = 1, 2$ ),  $\mathbf{a}$ ,  $\mathbf{a}^*$ , and  $\mathbf{T}^*$ , begin by noting that  $\mathbf{v}^*$ , the velocity of the mass center of  $R$ , is given by

$$\mathbf{v}^* = u_2 \mathbf{s}_1 \times (L\mathbf{r}_1) = u_2 L \sin \beta \mathbf{r}_3 \tag{27}$$

so that

$$\mathbf{v}_1^* = 0 \quad \mathbf{v}_2^* = L \sin \beta \mathbf{r}_3 \quad (28)$$

while

$$\mathbf{a}^* = \frac{d\mathbf{v}^*}{dt} = L \sin \beta (\dot{u}_2 \mathbf{r}_3 - u_2^2 \mathbf{s}_2) \quad (29)$$

Next, write

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (u_1 \mathbf{r}_1 + u_2 q_1 \sin \beta \mathbf{r}_3) \\ &= \dot{u}_1 \mathbf{r}_1 + (\dot{u}_2 q_1 + u_2 \dot{q}_1) \sin \beta \mathbf{r}_3 + (u_2 \mathbf{s}_1) \times (u_1 \mathbf{r}_1 + u_2 q_1 \sin \beta \mathbf{r}_3) \\ &= \dot{u}_1 \mathbf{r}_1 + [(\dot{u}_2 q_1 + 2u_1 u_2) \mathbf{r}_3 - u_2^2 q_1 \mathbf{s}_2] \sin \beta \end{aligned} \quad (30)$$

and express  $\omega_j$ ,  $\alpha_j$ , and  $I_j$  ( $j = 1, 2, 3$ ), needed for substitution into Eq. (10), as

$$\omega_1 = \boldsymbol{\omega} \cdot \mathbf{r}_1 = u_2 \mathbf{s}_1 \cdot \mathbf{r}_1 = u_2 \cos \beta \quad \omega_2 = -u_2 \sin \beta \quad \omega_3 = 0 \quad (31)$$

$$\alpha_1 = \dot{\omega}_1 = \dot{u}_2 \cos \beta \quad \alpha_2 = -\dot{u}_2 \sin \beta \quad \alpha_3 = 0 \quad (32)$$

$$I_1 = 0 \quad I_2 = \frac{ML^2}{3} \quad I_3 = \frac{ML^2}{3} \quad (33)$$

so that, from Eq. (10),

$$\mathbf{T}^* = \frac{ML^2}{3} \dot{u}_2 \sin \beta \mathbf{r}_2 + \frac{ML^2}{2} u_2^2 \sin \beta \cos \beta \mathbf{r}_3 \quad (34)$$

Finally, substitute into Eqs. (26) to obtain

$$\begin{aligned} F_1^* &= -m \mathbf{r}_1 \cdot \{ \dot{u}_1 \mathbf{r}_1 + [(\dot{u}_2 q_1 + 2u_1 u_2) \mathbf{r}_3 - u_2^2 q_1 \mathbf{s}_2] \sin \beta \} \\ &= -m(\dot{u}_1 - u_2^2 q_1 \sin^2 \beta) \end{aligned} \quad (35)$$

$$F_2^* = - \left[ \left( m q_1^2 + \frac{4ML^2}{3} \right) \dot{u}_2 + 2m q_1 u_1 u_2 \right] \sin^2 \beta \quad (36)$$

Equations (35) and (36) furnish expressions for the generalized inertia forces that correspond to the generalized active forces  $F_1$  and  $F_2$  given in Eqs. (4.9.6) and (4.9.7), respectively. To form the generalized inertia forces  $F_3^*$ ,  $F_4^*$ , and  $F_5^*$  corresponding to the generalized active forces  $F_3$ ,  $F_4$ , and  $F_5$  in Eqs. (4.9.19)–(4.9.21), all one needs to do is use Eqs. (26) with  $r = 3, 4, 5$  and with  $\mathbf{v}_r$ ,  $\boldsymbol{\omega}_r$ , and  $\mathbf{v}_r^*$ , ( $r = 3, 4, 5$ ) as given in Table 4.9.1, while  $\mathbf{a}^*$ ,  $\mathbf{a}$ ,



and  $\mathbf{T}^*$  are given by Eqs. (29), (30), and (34), respectively, as heretofore. Thus one finds that

$$F_3^* = -m\dot{u}_1 \cos \beta \quad (37)$$

$$F_4^* = \left[ \left( mq_1^2 + \frac{4ML^2}{3} \right) \dot{u}_2 + 2mq_1 u_1 u_2 \right] \sin \beta \cos \beta \quad (38)$$

$$F_5^* = -m(\dot{u}_2 q_1 + 2u_1 u_2) \sin \beta \quad (39)$$

Finally, by proceeding similarly, one can form expressions for  $F_6^*, \dots, F_9^*$  corresponding, respectively, to the generalized active forces in Eqs. (4.9.25)–(4.9.28). For example,  $F_9^*$  is thus found to be given by

$$F_9^* = mu_2^2 q_1 \sin \beta \cos \beta \quad (40)$$

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CHAPTER  
**FIVE**

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## ENERGY FUNCTIONS

The use of potential energy functions and kinetic energy functions sometimes enables one to construct integrals of equations of motion (see Secs. 7.1 and 7.2). In addition, potential energy functions can be helpful when one seeks to form expressions for generalized active forces, and expressions for generalized inertia forces can be formed with the aid of kinetic energy functions. Hence, familiarity with these functions is certainly desirable. However, since one can readily formulate equations of motion and extract information from such equations without invoking energy concepts, one need not master the material in the present chapter before moving on to Chapter 6.

### 5.1 POTENTIAL ENERGY

If  $S$  is a holonomic system (see Sec. 2.13) possessing generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10) and generalized speeds  $u_1, \dots, u_n$  (see Sec. 2.12) in a reference frame  $A$ , and the generalized speeds are defined as

$$u_r \triangleq \dot{q}_r \quad (r = 1, \dots, n) \quad (1)$$

then there may exist functions  $V$  of  $q_1, \dots, q_n$  and the time  $t$  that satisfy *all* of the equations

$$F_r = - \frac{\partial V}{\partial q_r} \quad (r = 1, \dots, n) \quad (2)$$

where  $F_1, \dots, F_n$  are generalized active forces for  $S$  in  $A$  (see Sec. 4.4) associated with  $u_1, \dots, u_n$ , respectively. Any such function  $V$  is called a *potential energy* of  $S$  in  $A$ . [One speaks of a potential energy, rather than *the* potential energy because, if  $V$  satisfies Eqs. (2), then  $V + C$ , where  $C$  is any function of  $t$ , also satisfies Eqs. (2) and is, therefore, a potential energy of  $S$  in  $A$ .]

When a potential energy  $V$  of  $S$  satisfies the equation

$$\frac{\partial V}{\partial t} = 0 \quad (3)$$

then  $\dot{V}$ , the total time-derivative of  $V$ , is given by

$$\dot{V} = - \sum_{r=1}^n F_r \dot{q}_r \quad (4)$$

It is by virtue of this fact that potential energy plays an important part in the construction of integrals of equations of motion, as will be shown in Sec. 7.2.

Given generalized active forces  $F_r$  ( $r = 1, \dots, n$ ) all of which can be regarded as functions of  $q_1, \dots, q_n$ , and  $t$  (but not of  $u_1, \dots, u_n$ ), one can either prove that  $V$  does not exist, or find  $V(q_1, \dots, q_n; t)$  explicitly, as follows: Determine whether or not *all* of the equations

$$\frac{\partial F_r}{\partial q_s} = \frac{\partial F_s}{\partial q_r} \quad (r, s = 1, \dots, n) \quad (5)$$

are satisfied. If one or more of Eqs. (5) are violated, then  $V$  does not exist; if all of Eqs. (5) are satisfied, then  $V$  exists and is given by

$$\begin{aligned} V = & \int_{\alpha_1}^{q_1} \frac{\partial}{\partial q_1} V(\zeta, \alpha_2, \dots, \alpha_n; t) d\zeta + \int_{\alpha_2}^{q_2} \frac{\partial}{\partial q_2} V(q_1, \zeta, \alpha_3, \dots, \alpha_n; t) d\zeta \\ & + \dots + \int_{\alpha_n}^{q_n} \frac{\partial}{\partial q_n} V(q_1, \dots, q_{n-1}, \zeta; t) d\zeta + C \end{aligned} \quad (6)$$

where  $\alpha_1, \dots, \alpha_n$  and  $C$  are *any* functions of  $t$ . [It is advantageous to set as many of  $\alpha_1, \dots, \alpha_n$  equal to zero as is possible without rendering any of the integrals in Eq. (6) improper.]

When  $S$  is holonomic and  $u_1, \dots, u_n$  are defined as

$$u_r \triangleq \sum_{(2.12.1) \ s=1}^n Y_{rs} \dot{q}_s + Z_r \quad (r = 1, \dots, n) \quad (7)$$

rather than as in Eqs. (1), so that

$$\dot{q}_s = \sum_{(2.14.5) \ r=1}^n W_{sr} u_r + X_s \quad (s = 1, \dots, n) \quad (8)$$

where  $W_{sr}$  and  $X_s$  are functions of  $q_1, \dots, q_n$ , and  $t$ , then Eqs. (2)–(4) give way to

$$F_r = - \sum_{s=1}^n \frac{\partial V}{\partial q_s} W_{sr} \quad (r = 1, \dots, n) \quad (9)$$

$$\frac{\partial V}{\partial t} + \sum_{s=1}^n \frac{\partial V}{\partial q_s} X_s = 0 \quad (10)$$

and

$$\dot{V} = - \sum_{r=1}^n F_r u_r \quad (11)$$

respectively. Under these circumstances, one can either prove that  $V$  does not exist, or find  $V(q_1, \dots, q_n; t)$  explicitly, as follows: Solve Eqs. (9) for  $\partial V / \partial q_s$  ( $s = 1, \dots, n$ ) and determine whether or not *all* of the equations

$$\frac{\partial}{\partial q_s} \left( \frac{\partial V}{\partial q_r} \right) = \frac{\partial}{\partial q_r} \left( \frac{\partial V}{\partial q_s} \right) \quad (r, s = 1, \dots, n) \quad (12)$$

are satisfied. If one or more of Eqs. (12) are violated, then  $V$  does not exist; if all of Eqs. (12) are satisfied, then  $V$  exists and can be found by using Eq. (6).

When  $S$  is a simple nonholonomic system possessing  $p$  degrees of freedom in  $A$  (see Sec. 2.13),  $u_1, \dots, u_n$  are defined as in Eqs. (1), and the motion constraint equations relating  $\dot{q}_{p+1}, \dots, \dot{q}_n$  to  $\dot{q}_1, \dots, \dot{q}_p$  are [this is a special case of Eqs. (2.13.1)]

$$\dot{q}_k = \sum_{r=1}^p C_{kr} \dot{q}_r + D_k \quad (k = p + 1, \dots, n) \quad (13)$$

where  $C_{kr}$  and  $D_k$  are functions of  $q_1, \dots, q_n$ , and  $t$ , then Eqs. (2), (3), and (4) are replaced with

$$\tilde{F}_r = - \left( \frac{\partial V}{\partial q_r} + \sum_{s=p+1}^n \frac{\partial V}{\partial q_s} C_{sr} \right) \quad (r = 1, \dots, p) \quad (14)$$

$$\frac{\partial V}{\partial t} + \sum_{s=p+1}^n \frac{\partial V}{\partial q_s} D_s = 0 \quad (15)$$

and

$$\dot{V} = - \sum_{r=1}^p \tilde{F}_r \dot{q}_r \quad (16)$$

respectively, whereas, when  $u_1, \dots, u_n$  are defined as in Eqs. (7), so that Eqs. (8) apply, while the motion constraint equations relating  $u_{p+1}, \dots, u_n$  to  $u_1, \dots, u_p$  are

$$u_k = \sum_{(2.13.1) r=1}^p A_{kr} u_r + B_k \quad (k = p + 1, \dots, n) \quad (17)$$

where  $A_{kr}$  and  $B_k$  are functions of  $q_1, \dots, q_n$ , and  $t$ , then Eqs. (2), (3), and (4) are replaced with

$$\tilde{F}_r = - \sum_{s=1}^n \frac{\partial V}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (18)$$

$$\frac{\partial V}{\partial t} + \sum_{s=1}^n \frac{\partial V}{\partial q_s} \left( X_s + \sum_{r=p+1}^n W_{sr} B_r \right) = 0 \quad (19)$$

and

$$\dot{V} = - \sum_{r=1}^p \tilde{F}_r u_r \quad (20)$$

respectively. In both cases, the procedure for either proving that  $V$  does not exist or finding  $V$  explicitly is more complicated than in the two cases considered previously, the underlying reason for this being that the  $n$  partial derivatives  $\partial V/\partial q_1, \dots, \partial V/\partial q_n$  needed in Eqs. (6) appear in only  $p$  equations, namely, Eqs. (14) or (18). What follows is a seven-step procedure for surmounting this hurdle.

**Step 1** Introduce  $m \triangleq n - p$  quantities  $f_1, \dots, f_m$  as

$$f_{s-p} \triangleq \frac{\partial V}{\partial q_s} \quad (s = p + 1, \dots, n) \quad (21)$$

and regard each of these as a function of  $q_1, \dots, q_n$ , and  $t$ , except when *both* of the following conditions are fulfilled for some value of  $r$ , say,  $r = i$ : (1) the generalized active force  $\tilde{F}_i$  is a function of  $q_i$  only; (2) the right-hand members of Eqs. (14) or (18) reduce to  $-\partial V/\partial q_i$ . In that event, regard each of  $f_1, \dots, f_m$  as a function of  $t$  and all of  $q_1, \dots, q_n$  except  $q_i$ . [Unless this is done, Eqs. (21) and the now applicable relationship  $\tilde{F}_i = -\partial V/\partial q_i$  lead to conflicting expressions for  $\partial^2 V/\partial q_s \partial q_i$  ( $s = p + 1, \dots, n$ ;  $s \neq i$ ), namely,  $\partial f_{s-p}/\partial q_i \neq 0$  and  $\partial \tilde{F}_i/\partial q_s = 0$ , respectively.]

**Step 2** In accordance with Eqs. (21), replace  $\partial V/\partial q_s$  with  $f_{s-p}$  ( $s = p + 1, \dots, n$ ) in Eqs. (14) or (18), and solve the resulting  $p$  equations for  $\partial V/\partial q_r$  ( $r = 1, \dots, p$ ).

**Step 3** Using the expressions obtained in Step 2 for  $\partial V/\partial q_r$  ( $r = 1, \dots, p$ ), form  $p(n - 1)$  expressions for  $\partial(\partial V/\partial q_r)/\partial q_j$  ( $r = 1, \dots, p$ ;  $j = 1, \dots, n$ ;  $j \neq r$ ). Referring to Eqs. (21), form the  $m(n - 1)$  equations  $\partial(\partial V/\partial q_s)/\partial q_j = \partial f_{s-p}/\partial q_j$  ( $s = p + 1, \dots, n$ ;  $j = 1, \dots, n$ ;  $j \neq s$ ). Substitute into Eqs. (12) to obtain  $n(n - 1)/2$  linear algebraic equations in the  $mn$  quantities  $\partial f_i/\partial q_j$  ( $i = 1, \dots, m$ ;  $j = 1, \dots, n$ ).

**Step 4** Identify an  $n(n - 1)/2 \times mn$  matrix  $[Z]$  and an  $n(n - 1)/2 \times 1$  matrix  $\{Y\}$  such that the set of equations written in Step 3 is equivalent to the matrix equation  $[Z] \{X\} = \{Y\}$ , where  $\{X\}$  is an  $mn \times 1$  matrix having  $\partial f_1/\partial q_1, \dots, \partial f_1/\partial q_n, \dots, \partial f_m/\partial q_1, \dots, \partial f_m/\partial q_n$  as successive elements.

**Step 5** Determine the rank  $\rho$  of  $[Z]$ . If  $\rho = n(n-1)/2$ , then  $V$  may exist, but cannot be found by the application of a straightforward procedure. If  $\rho \neq n(n-1)/2$ , use any  $\rho$  rows of  $[Z]$ , hereafter called independent rows, to express each of the remaining rows of  $[Z]$ , hereafter called the dependent rows, as a weighted, linear combination of the  $\rho$  independent rows; and solve the resulting set of equations simultaneously to determine the weighting factors.

**Step 6** Express each element of  $\{Y\}$  corresponding to a dependent row of  $[Z]$  as a weighted, linear combination of the  $\rho$  elements of  $\{Y\}$  corresponding to the independent rows of  $[Z]$ , using the weighting factors found in Step 5, and solve the resulting set of equations for  $f_1, \dots, f_m$ . If this cannot be done uniquely, or if one or more of  $f_1, \dots, f_m$  turn out to be functions of a generalized coordinate of which they should be independent in accordance with Step 1, then a potential energy  $V$  of  $S$  in  $A$  does not exist.

**Step 7** Substitute the functions  $f_1, \dots, f_m$  found in Step 6 into Eqs. (21) and into the expressions for  $\partial V/\partial q_1, \dots, \partial V/\partial q_s$  formed in Step 2, thus obtaining expressions for  $\partial V/\partial q_1, \dots, \partial V/\partial q_n$  as explicit functions of  $q_1, \dots, q_n$ , and  $t$ . Finally, form  $V$  in accordance with Eq. (6).

**Derivations** Multiplication of both sides of Eqs. (18) with  $u_r$  and subsequent summation yields

$$\sum_{r=1}^p \tilde{F}_r u_r = - \sum_{r=1}^p \left[ \sum_{s=1}^n \frac{\partial V}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \right] u_r \quad (22)$$

or, equivalently,

$$- \sum_{r=1}^p \tilde{F}_r u_r = \sum_{s=1}^n \left[ \frac{\partial V}{\partial q_s} \sum_{r=1}^p \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) u_r \right] \quad (23)$$

Now,

$$\begin{aligned} \sum_{r=1}^p \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) u_r &= \sum_{r=1}^p W_{sr} u_r + \sum_{k=p+1}^n W_{sk} \sum_{r=1}^p A_{kr} u_r \\ &= \sum_{r=1}^p W_{sr} u_r + \sum_{k=p+1}^n W_{sk} (u_k - B_k) \quad (17) \\ &= \sum_{r=1}^p W_{sr} u_r - \sum_{k=p+1}^n W_{sk} B_k \\ &= \dot{q}_s - \left( X_s + \sum_{k=p+1}^n W_{sk} B_k \right) \quad (s = 1, \dots, n) \quad (24) \end{aligned}$$

Consequently,

$$\begin{aligned}
 - \sum_{r=1}^p \tilde{F}_r u_r &= \sum_{(23) s=1}^n \frac{\partial V}{\partial q_s} \left[ \dot{q}_s - \left( X_s + \sum_{k=p+1}^n W_{sk} B_k \right) \right] \\
 &\quad (24) \\
 &= \dot{V} - \left[ \frac{\partial V}{\partial t} + \sum_{s=1}^n \frac{\partial V}{\partial q_s} \left( X_s + \sum_{k=p+1}^n W_{sk} B_k \right) \right] \\
 &\stackrel{(19)}{=} \dot{V} \quad (25)
 \end{aligned}$$

which is Eq. (20).

One can obtain Eqs. (1) by taking  $W_{sr} = \delta_{sr}$ , the Kronecker delta, and  $X_s = 0$  ( $r, s = 1, \dots, n$ ) in Eqs. (8); and, setting  $A_{kr} = C_{kr}$  and  $B_k = D_k$  ( $r = 1, \dots, p$ ;  $k = p+1, \dots, n$ ), one then finds that Eqs. (17)–(20) lead to Eqs. (13)–(16), respectively. When  $p = n$ , in which event Eqs. (17) drop out of the picture, then Eqs. (18)–(20) become Eqs. (9)–(11), respectively. Finally, when Eqs. (8) reduce to Eqs. (1) and  $p = n$ , then Eqs. (18)–(20) reduce to Eqs. (2)–(4), respectively.

The rationale underlying the seven-step procedure for the construction of potential energy functions is the following. Step 1 is taken in recognition of the fact that Eqs. (18) or (14) form a set of  $p$  equations in the  $n$  partial derivatives  $\partial V/\partial q_1, \dots, \partial V/\partial q_n$ , so that, since  $n = p + m$ ,  $m$  additional relationships are required for the determination of all of these  $n$  partial derivatives. In Step 2, the task begun in Step 1, that is, the constructing of a set of expressions for the partial derivatives  $\partial V/\partial q_1, \dots, \partial V/\partial q_n$ , is brought to completion. Step 3 consists of imposing requirements that must be satisfied in order that  $V$  possess continuous first partial derivatives. Steps 4–6 allow one to determine  $f_i$  ( $i = 1, \dots, m$ ) by exploiting the fact† that the matrix equation  $[Z] \{X\} = \{Y\}$  can be solved uniquely for  $\{X\}$  if and only if the rank of  $[Z]$  is equal to that of the matrix  $[[Z] \vdots \{Y\}]$ . Finally, in Step 7 the desired expression for  $V$  is formed. [Partial differentiations of Eq. (6) with respect to  $q_1, \dots, q_n$  show that this equation is a generally valid relationship between a function  $V$  of  $q_1, \dots, q_n$ , and  $t$  and the partial derivatives  $\partial V/\partial q_1, \dots, \partial V/\partial q_n$  when Eqs. (12) are satisfied.]

**Example** Suppose that the system  $S$  formed by the particle  $P_1$  and the sharp-edged circular disk  $D$  considered in the example in Sec. 2.13 and shown in Fig. 2.13.1 is subjected to the action of a contact force  $\mathbf{K}$  applied to  $P_1$ , with  $\mathbf{K}$  given by

$$\mathbf{K} = k\mathbf{e}_x - \frac{k}{L} \mathbf{p} \cdot \mathbf{e}_y \mathbf{e}_y \quad (26)$$

where  $k$  is a constant,  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are unit vectors directed as shown in Fig. 2.13.1, and  $\mathbf{p}$  is the position vector from point  $O$  to  $P_1$ . Furthermore, let

† M. H. Protter and C. B. Morrey, Jr., *Modern Mathematical Analysis* (Addison-Wesley, Reading, Mass., 1966), p. 300.

$m_1$  and  $m_2$  be the masses of  $P_1$  and  $D$ , respectively; regard the rod  $R$  connecting  $P_1$  and  $D$  as having a negligible mass; assume that  $Y$ , the axis of rotation of  $B$ , is vertical; and, designating as  $E$  a rigid body in which  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ , and  $\mathbf{e}_z \triangleq \mathbf{e}_x \times \mathbf{e}_y$  are fixed, define generalized speeds  $u_1$ ,  $u_2$ , and  $u_3$  as

$$u_1 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_x \quad u_2 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_y \quad u_3 \triangleq {}^B\boldsymbol{\omega}^E \cdot \mathbf{e}_z \quad (27)$$

so that, in accordance with Eqs. (2.12.8),

$$\dot{q}_1 = u_1 c_3 - u_2 s_3 \quad \dot{q}_2 = u_1 s_3 + u_2 c_3 \quad \dot{q}_3 = u_3 \quad (28)$$

Then  $S$  is subject to the motion constraint

$$u_3 \underset{(2.13.12)}{=} -\frac{u_2}{L} \quad (29)$$

arising from the requirement that  ${}^B\mathbf{v}^{D^*} \cdot \mathbf{e}_y$  be equal to zero, where  ${}^B\mathbf{v}^{D^*}$  is the velocity in  $B$  of the center  $D^*$  of  $D$ , and Eqs. (28) and (29) play the roles of Eqs. (8) and (17), respectively; that is,

$$n = 3 \quad m = 1 \quad p = n - m = 2 \quad (30)$$

and

$$W_{11} = c_3 \quad W_{12} = -s_3 \quad W_{13} = 0 \quad (31)$$

$$W_{21} = s_3 \quad W_{22} = c_3 \quad W_{23} = 0 \quad (32)$$

$$W_{31} = 0 \quad W_{32} = 0 \quad W_{33} = 1 \quad (33)$$

$$X_1 = X_2 = X_3 = 0 \quad (34)$$

while

$$A_{31} = 0 \quad A_{32} = -\frac{1}{L} \quad B_3 = 0 \quad (35)$$

Moreover, the generalized active forces  $\tilde{F}_1$  and  $\tilde{F}_2$  for  $S$  in  $A$  are given by

$$\tilde{F}_r \underset{(4.4.1)}{=} {}^A\tilde{\mathbf{v}}_r^{P_1} \cdot (\mathbf{K} + m_1 g \mathbf{z}) + {}^A\tilde{\mathbf{v}}_r^{D^*} \cdot (m_2 g \mathbf{z}) \quad (r = 1, 2) \quad (36)$$

where  $\mathbf{z}$  is a unit vector directed vertically downward; the partial velocities  ${}^A\tilde{\mathbf{v}}_r^{P_1}$  and  ${}^A\tilde{\mathbf{v}}_r^{D^*}$  ( $r = 1, 2$ ) are available in Eqs. (2.14.25) and (2.14.31), respectively. Referring to Eq. (26) and noting that the position vector from  $O$  to  $P_1$  can be written

$$\mathbf{p} = (q_1 c_3 + q_2 s_3)\mathbf{e}_x - (q_1 s_3 - q_2 c_3)\mathbf{e}_y \quad (37)$$

while

$$\mathbf{z} = -(s_3 \mathbf{e}_x + c_3 \mathbf{e}_y) \quad (38)$$

one thus has

$$\tilde{F}_1 \underset{(36)}{=} k - (m_1 + m_2)gs_3 \quad \tilde{F}_2 \underset{(36)}{=} \frac{k}{L}(q_1 s_3 - q_2 c_3) - m_1 g c_3 \quad (39)$$



A potential energy  $V$  of  $S$  now will be found by following the seven-step procedure after noting that substitution from Eqs. (31)–(33) and (39) into Eqs. (18) yields

$$k - (m_1 + m_2)gs_3 = -\left(\frac{\partial V}{\partial q_1}c_3 + \frac{\partial V}{\partial q_2}s_3\right) \quad (40)$$

$$\frac{k}{L}(q_1s_3 - q_2c_3) - m_1gc_3 = \frac{\partial V}{\partial q_1}s_3 - \frac{\partial V}{\partial q_2}c_3 + \frac{1}{L}\frac{\partial V}{\partial q_3} \quad (41)$$

In accordance with Eqs. (21),  $f_1$  is introduced as

$$f_1 \triangleq \frac{\partial V}{\partial q_3} \quad (42)$$

and is regarded as a function of  $q_1$ ,  $q_2$ , and  $q_3$  because, although  $\tilde{F}_1$  [see Eqs. (39)] is a function of  $q_3$  only, the right-hand member of Eq. (40) is not  $-\partial V/\partial q_3$ .

**Step 2** Elimination of  $\partial V/\partial q_3$  from Eq. (41) with the aid of Eq. (42), and simultaneous solution of the resulting equation and Eq. (40) for  $\partial V/\partial q_1$  and  $\partial V/\partial q_2$ , yields

$$\frac{\partial V}{\partial q_1} = -kc_3 + \frac{k}{L}(q_1s_3 - q_2c_3)s_3 + gm_2s_3c_3 - \frac{f_1}{L}s_3 \quad (43)$$

$$\frac{\partial V}{\partial q_2} = -ks_3 - \frac{k}{L}(q_1s_3 - q_2c_3)c_3 + g(m_1 + m_2s_3^2) + \frac{f_1}{L}c_3 \quad (44)$$

**Step 3** Differentiations of Eqs. (42)–(44) yield the following mixed partial derivatives of  $V$ :

$$\frac{\partial^2 V}{\partial q_1 \partial q_2} = -\frac{k}{L}s_3c_3 + \frac{1}{L}\frac{\partial f_1}{\partial q_1}c_3 \quad (45)$$

$$\frac{\partial^2 V}{\partial q_2 \partial q_1} = -\frac{k}{L}c_3s_3 - \frac{1}{L}\frac{\partial f_1}{\partial q_2}s_3 \quad (46)$$

$$\frac{\partial^2 V}{\partial q_2 \partial q_3} = \frac{\partial f_1}{\partial q_2} \quad (47)$$

$$\begin{aligned} \frac{\partial^2 V}{\partial q_3 \partial q_2} = & -kc_3 - \frac{k}{L}[q_1(c_3^2 - s_3^2) + 2q_2s_3c_3] \\ & + 2gm_2s_3c_3 + \frac{1}{L}\left(\frac{\partial f_1}{\partial q_3}c_3 - f_1s_3\right) \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{\partial^2 V}{\partial q_3 \partial q_1} = & ks_3 + \frac{k}{L}[2q_1s_3c_3 - q_2(c_3^2 - s_3^2)] + gm_2(c_3^2 - s_3^2) \\ & - \frac{1}{L}\left(\frac{\partial f_1}{\partial q_3}s_3 + f_1c_3\right) \end{aligned} \quad (49)$$

$$\frac{\partial^2 V}{\partial q_1 \partial q_3} = \frac{\partial f_1}{\partial q_1} \quad (50)$$

Equations (12) thus lead to the following three equations in  $\partial f_1/\partial q_1$ ,  $\partial f_1/\partial q_2$ ,  $\partial f_1/\partial q_3$ :

$$c_3 \frac{\partial f_1}{\partial q_1} + s_3 \frac{\partial f_1}{\partial q_2} = 0 \quad (51)$$

$$\begin{aligned} \frac{\partial f_1}{\partial q_2} - \frac{c_3}{L} \frac{\partial f_1}{\partial q_3} = & -\frac{s_3}{L} f_1 + 2gm_2 s_3 c_3 - kc_3 \\ & - \frac{k}{L} [q_1(c_3^2 - s_3^2) + 2q_2 s_3 c_3] \end{aligned} \quad (52)$$

$$\begin{aligned} \frac{\partial f_1}{\partial q_1} + \frac{s_3}{L} \frac{\partial f_1}{\partial q_3} = & -\frac{c_3}{L} f_1 + gm_2(c_3^2 - s_3^2) + ks_3 \\ & + \frac{k}{L} [2q_1 s_3 c_3 - q_2(c_3^2 - s_3^2)] \end{aligned} \quad (53)$$

**Step 4** Inspection of Eqs. (51)–(53) reveals that this set of equations is equivalent to  $[Z]\{X\} = \{Y\}$  if  $\{X\}$ ,  $\{Y\}$ , and  $[Z]$  are defined as

$$\{X\} \triangleq \begin{Bmatrix} \frac{\partial f_1}{\partial q_1} \\ \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_1}{\partial q_3} \end{Bmatrix} \quad (54)$$

$$\{Y\} \triangleq \begin{Bmatrix} 0 \\ -\frac{s_3}{L} f_1 + 2gm_2 s_3 c_3 - kc_3 - \frac{k}{L} [q_1(c_3^2 - s_3^2) + 2q_2 s_3 c_3] \\ -\frac{c_3}{L} f_1 + gm_2(c_3^2 - s_3^2) + ks_3 + \frac{k}{L} [2q_1 s_3 c_3 - q_2(c_3^2 - s_3^2)] \end{Bmatrix} \quad (55)$$

and

$$[Z] \triangleq \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 1 & -\frac{c_3}{L} \\ 1 & 0 & \frac{s_3}{L} \end{bmatrix} \quad (56)$$

**Step 5** The matrix  $[Z]$  is singular, but it possesses a nonvanishing determinant of order 2. Hence,  $\rho = 2$ . Selecting the first two rows of  $[Z]$  as the ones to be treated as independent, one can express the third row as

$$[1 \quad 0 \quad s_3/L] = w_1[c_3 \quad s_3 \quad 0] + w_2[0 \quad 1 \quad -c_3/L] \quad (57)$$

where  $w_1$  and  $w_2$  are weighting factors. Equating the first elements on the right-hand and left-hand sides of Eq. (57), one finds that  $w_1 = 1/c_3$ , and equating the second elements then leads to  $w_2 = -s_3/c_3$ .

**Step 6** Expressing  $Y_3$ , the element in the third row of  $\{Y\}$  in Eq. (55), as  $w_1 Y_1 + w_2 Y_2$ , where  $Y_1$  and  $Y_2$  are the elements in the first two rows of  $\{Y\}$ , one has

$$\begin{aligned} & -\frac{c_3}{L}f_1 + gm_2(c_3^2 - s_3^2) + ks_3 + \frac{k}{L}[2q_1s_3c_3 - q_2(c_3^2 - s_3^2)] \\ & = -\frac{s_3}{c_3} \left\{ -\frac{s_3}{L}f_1 + 2gm_2s_3c_3 - kc_3 - \frac{k}{L}[q_1(c_3^2 - s_3^2) + 2q_2s_3c_3] \right\} \quad (58) \end{aligned}$$

and, solving this equation for  $f_1$ , one finds that

$$f_1 = m_2gLc_3 + k(q_1s_3 - q_2c_3) \quad (59)$$

**Step 7** Substituting  $f_1$  as given in Eq. (59) into Eqs. (42)–(44), one arrives at

$$\frac{\partial V}{\partial q_1 (43)} = -kc_3 \quad (60)$$

$$\frac{\partial V}{\partial q_2 (44)} = -ks_3 + g(m_1 + m_2) \quad (61)$$

$$\frac{\partial V}{\partial q_3 (42)} = m_2gLc_3 + k(q_1s_3 - q_2c_3) \quad (62)$$

and, proceeding in accordance with Eq. (6) after setting  $\alpha_1 = \alpha_2 = \alpha_3 = C = 0$ , one can thus write

$$\begin{aligned} V = & \int_0^{q_1} \underset{(60)}{[-k \cos(0)]} d\zeta + \int_0^{q_2} \underset{(61)}{[-k \sin(0) + g(m_1 + m_2)]} d\zeta \\ & + \int_0^{q_3} \underset{(62)}{[m_2gL \cos \zeta + k(q_1 \sin \zeta - q_2 \cos \zeta)]} d\zeta \quad (63) \end{aligned}$$

so that, after performing the indicated integrations, one has

$$V = -k(q_1c_3 + q_2s_3) + g[(m_1 + m_2)q_2 + m_2Ls_3] \quad (64)$$

For the problem at hand, Eq. (19) reduces to

$$\frac{\partial V}{\partial t} = 0 \quad (65)$$

by virtue of Eqs. (34) and the last of Eqs. (35). Since  $V$  as given by Eq. (64) satisfies Eq. (65), Eq. (20) should be satisfied. To see that this is, in fact, the case, note that

$$\begin{aligned} \dot{V} = & \underset{(64)}{-k(\dot{q}_1 c_3 - q_1 \dot{q}_3 s_3 + \dot{q}_2 s_3 + q_2 \dot{q}_3 c_3)} \\ & + g[(m_1 + m_2)\dot{q}_2 + m_2 L \dot{q}_3 c_3] \end{aligned} \quad (66)$$

and that, when Eqs. (28) are used to eliminate  $\dot{q}_1$ ,  $\dot{q}_2$ , and  $\dot{q}_3$  from Eq. (66), one obtains, with the aid of Eq. (29),

$$\dot{V} = \underset{(66)}{-[k - (m_1 + m_2)gs_3]u_1 - \left[ \frac{k}{L}(q_1 s_3 - q_2 c_3) - m_1 g c_3 \right]u_2} \quad (67)$$

or, in view of Eqs. (39),

$$\dot{V} = -(\tilde{F}_1 u_1 + \tilde{F}_2 u_2) \quad (68)$$

in agreement with Eq. (20).

To illustrate the possibility of nonexistence of a potential energy, suppose that the disk  $D$  is replaced with a particle  $P_2$  of mass  $m_2$ , and let  $P_2$  be free to slide in plane  $B$  (see Fig. 2.13.1), so that the motion constraint expressed by Eq. (29) no longer applies and the system  $S$  formed by  $P_1$  and  $P_2$  is a holonomic system possessing three degrees of freedom in  $A$ . The associated generalized active forces then are given by

$$F_r = \underset{(4.4.2)}{{}^A \mathbf{v}_r^{P_1} \cdot (\mathbf{K} + m_1 g \mathbf{z}) + {}^A \mathbf{v}_r^{P_2} \cdot (m_2 g \mathbf{z})} \quad (r = 1, 2, 3) \quad (69)$$

where the partial velocities  ${}^A \mathbf{v}_r^{P_1}$  ( $r = 1, 2, 3$ ) are available in Eqs. (2.14.20) and the partial velocities  ${}^A \mathbf{v}_r^{P_2}$  ( $r = 1, 2, 3$ ) are [see Eqs. (2.14.30)]

$${}^A \mathbf{v}_1^{P_2} = \mathbf{e}_x \quad {}^A \mathbf{v}_2^{P_2} = \mathbf{e}_y \quad {}^A \mathbf{v}_3^{P_2} = L \mathbf{e}_y \quad (70)$$

Consequently [see Eq. (26) for  $\mathbf{K}$ , Eq. (37) for  $\mathbf{p}$ , and Eq. (38) for  $\mathbf{z}$ ],

$$F_1 = k - (m_1 + m_2)gs_3 \quad (71)$$

$$F_2 = \frac{k}{L}(q_1 s_3 - q_2 c_3) - (m_1 + m_2)gc_3 \quad (72)$$

$$F_3 = -m_2 g L c_3 \quad (73)$$

and Eqs. (9) can be written, with the aid of Eqs. (31)–(33), as

$$k - (m_1 + m_2)gs_3 = - \left( \frac{\partial V}{\partial q_1} c_3 + \frac{\partial V}{\partial q_2} s_3 \right) \quad (74)$$

$$\frac{k}{L}(q_1 s_3 - q_2 c_3) - (m_1 + m_2)gc_3 = - \left( - \frac{\partial V}{\partial q_1} s_3 + \frac{\partial V}{\partial q_2} c_3 \right) \quad (75)$$

$$-m_2 g L c_3 = - \frac{\partial V}{\partial q_3} \quad (76)$$

Solving these equations for  $\partial V/\partial q_r$  ( $r = 1, 2, 3$ ), one has

$$\frac{\partial V}{\partial q_1} = -kc_3 + \frac{k}{L}(q_1s_3 - q_2c_3)s_3 \quad (77)$$

$$\frac{\partial V}{\partial q_2} = -ks_3 - \frac{k}{L}(q_1s_3 - q_2c_3)c_3 + (m_1 + m_2)g \quad (78)$$

$$\frac{\partial V}{\partial q_3} = m_2gLc_3 \quad (79)$$

and differentiation of these relationships yields

$$\frac{\partial^2 V}{\partial q_1 \partial q_2} = -\frac{k}{L}s_3c_3 \quad \frac{\partial^2 V}{\partial q_2 \partial q_1} = -\frac{k}{L}c_3s_3 \quad (80)$$

$$\frac{\partial^2 V}{\partial q_2 \partial q_3} = 0 \quad \frac{\partial^2 V}{\partial q_3 \partial q_2} = -kc_3 - \frac{k}{L}[q_1(c_3^2 - s_3^2) + 2q_2s_3c_3] \quad (81)$$

$$\frac{\partial^2 V}{\partial q_3 \partial q_1} = ks_3 + \frac{k}{L}[2q_1s_3c_3 - q_2(c_3^2 - s_3^2)] \quad \frac{\partial^2 V}{\partial q_1 \partial q_3} = 0 \quad (82)$$

Hence, unless  $k = 0$ ,

$$\frac{\partial^2 V}{\partial q_2 \partial q_3} \neq \frac{\partial^2 V}{\partial q_3 \partial q_2} \quad (83)$$

and

$$\frac{\partial^2 V}{\partial q_3 \partial q_1} \neq \frac{\partial^2 V}{\partial q_1 \partial q_3} \quad (84)$$

so that not all of Eqs. (12) are satisfied. Therefore, there exists no potential energy.

## 5.2 POTENTIAL ENERGY CONTRIBUTIONS

Referring to Sec. 5.1, divide the set of all contact and/or distance forces contributing to  $\tilde{F}_r$  ( $r = 1, \dots, p$ ) into subsets  $\alpha, \beta, \dots$  associated with particular sets of contact and/or distance forces. Furthermore, let  $(\tilde{F}_r)_\alpha, (\tilde{F}_r)_\beta, \dots$  denote the contributions of  $\alpha, \beta, \dots$  respectively, to  $\tilde{F}_r$  ( $r = 1, \dots, p$ ), and let  $V_\alpha, V_\beta, \dots$  be functions of  $q_1, \dots, q_n$ , and  $t$  such that Eqs. (5.1.18) are satisfied when  $\tilde{F}_r$  and  $V$  are replaced with  $(\tilde{F}_r)_\alpha$  and  $V_\alpha$ , respectively, and similarly for  $\beta, \gamma, \dots$ . The functions  $V_\alpha, V_\beta, \dots$  are called *potential energy contributions* of  $\alpha, \beta, \dots$  for  $S$ , and the function  $V$  of  $q_1, \dots, q_n$ , and  $t$  defined as

$$V \triangleq V_\alpha + V_\beta + \dots \quad (1)$$

is a potential energy of  $S$  (see Sec. 5.1).

Considering the set  $\gamma$  of all gravitational forces exerted on particles of  $S$  by the Earth  $E$ , and assuming that these forces can be treated as in Sec. 4.8, let

$$V_\gamma \triangleq -Mg\mathbf{k} \cdot \mathbf{p}^* \quad (2)$$

where  $M$  is the total mass of  $S$ ,  $g$  is the local gravitational acceleration,  $\mathbf{k}$  is a unit vector locally directed vertically downward, and  $\mathbf{p}^*$  is the position vector from any point fixed in  $E$  to  $S^*$ . Then  $V_\gamma$  is a potential energy contribution of  $\gamma$  for  $S$ .

When one end of a spring is fixed in a reference frame  $A$  while the other end is attached to a particle of  $S$ , or when a spring connects two particles of  $S$  (the spring, itself, being in neither case a part of  $S$ ), let  $\sigma$  be the set of forces exerted by the spring on particles of  $S$ , and define  $V_\sigma$  as

$$V_\sigma \triangleq \int_0^x f(\zeta) d\zeta \quad (3)$$

where  $x$  is a function of  $q_1, \dots, q_n$ , and  $t$  that measures the *extension* of the spring, that is, the difference between the spring's current length and the spring's natural length;  $f(x)$  defines the spring's elastic characteristics. For instance,  $f(x)$  may be given by

$$f(x) = kx \quad (4)$$

where  $k$  is a constant called the *spring constant* or *spring modulus*. Under these circumstances, the spring is said to be a *linear* spring, and Eq. (3) leads to

$$V_\sigma = \frac{1}{2}kx^2 \quad (5)$$

In any event,  $V_\sigma$  as defined in Eqs. (3) [and, hence,  $V_\sigma$  as given in Eq. (5)] is a potential energy contribution of  $\sigma$  for  $S$ .

Equations (3)–(5) apply also when one end of a torsion spring is fixed in a reference frame  $A$  while the other end is attached to a rigid body  $B$  belonging to  $S$  and free to rotate relative to  $A$  about an axis fixed in both  $A$  and  $B$ , or when a torsion spring connects two rigid bodies belonging to  $S$  and free to rotate relative to each other about an axis fixed in both bodies. Under these circumstances,  $x$  measures the rotational deformation of the spring.

**Derivations** It follows from the definitions of  $(\tilde{F}_r)_\alpha$ ,  $(\tilde{F}_r)_\beta, \dots$  and  $V_\alpha, V_\beta, \dots$  that

$$\tilde{F}_r = (\tilde{F}_r)_\alpha + (\tilde{F}_r)_\beta + \dots \quad (r = 1, \dots, p) \quad (6)$$

and that

$$(\tilde{F}_r)_\alpha = - \sum_{s=1}^n \frac{\partial V_\alpha}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (7)$$

$$(\tilde{F}_r)_\beta = - \sum_{s=1}^n \frac{\partial V_\beta}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (8)$$

and so forth. Consequently,

$$(\tilde{F}_r)_\alpha + (\tilde{F}_r)_\beta + \cdots = - \sum_{s=1}^n \left[ \frac{\partial}{\partial q_s} (V_\alpha + V_\beta + \cdots) \right] \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (7,8)$$

$$(r = 1, \dots, p) \quad (9)$$

which reduces to Eq. (5.1.18) when Eqs. (6) and (1) are brought into play. Hence,  $V$  as defined in Eq. (1) is a potential energy of  $S$ .

In order to show that  $V_\gamma$  and  $V_\sigma$  are potential energy contributions of  $\gamma$  and  $\sigma$ , respectively, it is necessary to make use of the following kinematical proposition. If  $\mathbf{p}$ , regarded as a function of  $q_1, \dots, q_n$ , and  $t$  in  $A$ , is the position vector from a point fixed in  $A$  to a point  $P$  of  $S$ , and  $\mathbf{v}$ , the velocity of  $P$  in  $A$ , is expressed as in Eq. (2.14.4), then

$$\tilde{\mathbf{v}}_r = \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (10)$$

To see this, note that

$$\begin{aligned} \tilde{\mathbf{v}}_r &= \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sr} + \sum_{k=p+1}^n \mathbf{v}_k A_{kr} \\ &= \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sr} + \sum_{k=p+1}^n \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sk} A_{kr} \quad (r = 1, \dots, p) \end{aligned} \quad (11)$$

(2.14.11)

which is equivalent to Eqs. (10).

Now consider  $V_\gamma$  as defined in Eq. (2). Partial differentiations with respect to  $q_s$  ( $s = 1, \dots, n$ ) give

$$\frac{\partial V_\gamma}{\partial q_s} = -Mg\mathbf{k} \cdot \frac{\partial \mathbf{p}^*}{\partial q_s} \quad (s = 1, \dots, n) \quad (12)$$

Hence,

$$\begin{aligned} - \sum_{s=1}^n \frac{\partial V_\gamma}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) &= Mg\mathbf{k} \cdot \sum_{s=1}^n \frac{\partial \mathbf{p}^*}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \\ &= Mg\mathbf{k} \cdot \tilde{\mathbf{v}}_r^* = (\tilde{F}_r)_\gamma \quad (r = 1, \dots, p) \end{aligned} \quad (13)$$

(10) (4.8.2)

which shows that Eqs. (5.1.18) are satisfied when  $\tilde{F}_r$  and  $V$  are replaced with  $(\tilde{F}_r)_\gamma$  and  $V_\gamma$ , respectively, and this means that  $V_\gamma$  is a potential energy contribution of  $\gamma$  for  $S$ .

As for  $V_\sigma$ , defined in Eq. (3), we begin once more by forming partial derivatives with respect to  $q_s$  ( $s = 1, \dots, n$ ), obtaining

$$\frac{\partial V_\sigma}{\partial q_s} = f(x) \frac{\partial x}{\partial q_s} \quad (s = 1, \dots, n) \quad (14)$$

(3)

whereupon we can write

$$-\sum_{s=1}^n \frac{\partial V_\sigma}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) = -f(x) \sum_{s=1}^n \frac{\partial x}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (14)$$

$$(r = 1, \dots, p) \quad (15)$$

Next, we construct an expression for  $(\tilde{F}_r)_\sigma$ , the contribution to  $\tilde{F}_r$  of the force exerted by a spring on a particle  $P$  of  $S$  when one end of the spring is attached to a point  $O$  fixed in  $A$ . To facilitate this task, we introduce a unit vector  $\mathbf{e}$  directed from  $O$  to  $P$ , and express  $\mathbf{p}$ , the position vector from  $O$  to  $P$ , as

$$\mathbf{p} = (L + x)\mathbf{e} \quad (16)$$

where  $L$  is the natural length of the spring. The holonomic partial velocities of  $P$  in  $A$  then can be written

$$\mathbf{v}_r = \sum_{s=1}^n \frac{\partial \mathbf{p}}{\partial q_s} W_{sr} = \sum_{s=1}^n \left[ \frac{\partial x}{\partial q_s} \mathbf{e} + (L + x) \frac{\partial \mathbf{e}}{\partial q_s} \right] W_{sr} \quad (r = 1, \dots, n) \quad (17)$$

and, with  $\mathbf{T}$ , the (tensile) force exerted on  $P$  by the spring, expressed as

$$\mathbf{T} = -f(x)\mathbf{e} \quad (18)$$

we find that  $(F_r)_\sigma$ , the contribution of  $\mathbf{T}$  to the *holonomic* generalized active force  $F_r$ , is given by

$$(F_r)_\sigma = \mathbf{v}_r \cdot \mathbf{T} = -f(x) \sum_{s=1}^n \frac{\partial x}{\partial q_s} W_{sr} \quad (r = 1, \dots, n) \quad (19)$$

because  $2\mathbf{e} \cdot \partial \mathbf{e} / \partial q_s = \partial(\mathbf{e} \cdot \mathbf{e}) / \partial q_s = \partial(1) / \partial q_s = 0$  ( $s = 1, \dots, n$ ); and  $(\tilde{F}_r)_\sigma$ , the contribution of  $\mathbf{T}$  to the *nonholonomic* generalized active force  $\tilde{F}_r$ , now can be formed as

$$\begin{aligned} (\tilde{F}_r)_\sigma &= (F_r)_\sigma + \sum_{k=p+1}^n (F_k)_\sigma A_{kr} \\ &= -f(x) \left( \sum_{s=1}^n \frac{\partial x}{\partial q_s} W_{sr} + \sum_{k=p+1}^n \sum_{s=1}^n \frac{\partial x}{\partial q_s} W_{sk} A_{kr} \right) \\ &= -f(x) \sum_{s=1}^n \frac{\partial x}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \end{aligned} \quad (20)$$

The right-hand members of Eqs. (20) and (15) are identical. Consequently,

$$(\tilde{F}_r)_\sigma = - \sum_{s=1}^n \frac{\partial V_\sigma}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (21)$$

This is precisely Eq. (5.1.18) when  $\tilde{F}_r$  and  $V$  are replaced with  $(\tilde{F}_r)_\sigma$  and  $V_\sigma$ , respectively, which means that  $V_\sigma$  is a potential energy contribution of  $\sigma$  for  $S$ . A parallel proof shows that this conclusion is also valid when the spring connects two particles of  $S$ .



**Examples** Referring to the example in Sec. 4.4, one can express  $\mathbf{p}^*$ , the position vector from  $O$  to the mass center of the system  $S$  formed by  $P_1$  and  $P_2$ , as

$$\mathbf{p}^* = \frac{m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)}{m_1 + m_2} \mathbf{t}_1 \quad (22)$$

Forming, with the aid of Eq. (2), a potential energy contribution  $V_\gamma$  of the set  $\gamma$  of gravitational forces acting on  $S$ , one has

$$V_\gamma = -[m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)]g \cos \theta \quad (23)$$

(2.22)

The extensions of the springs  $\sigma_1$  and  $\sigma_2$  are  $q_1$  and  $q_2 - q_1$ , respectively. Hence, if potential energy contributions of the forces exerted by  $\sigma_1$  and  $\sigma_2$  are denoted by  $V_{\sigma_1}$  and  $V_{\sigma_2}$ , respectively, then, in accordance with Eq. (5),

$$V_{\sigma_1} = \frac{1}{2}k_1q_1^2 \quad V_{\sigma_2} = \frac{1}{2}k_2(q_2 - q_1)^2 \quad (24)$$

Now consider the function  $V$  defined as

$$\begin{aligned} V &\triangleq V_\gamma + V_{\sigma_1} + V_{\sigma_2} \\ &= -[m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)]g \cos \theta \\ &\quad + \frac{1}{2}[k_1q_1^2 + k_2(q_2 - q_1)^2] \end{aligned} \quad (25)$$

(23,24)

Since gravitational forces and forces exerted on  $P_1$  and  $P_2$  by  $\sigma_1$  and  $\sigma_2$  are the only forces contributing to generalized active forces for  $S$  in  $A$ ,  $V$  as given by Eq. (25) is a potential energy of  $S$  [see Eq. (1)]. It follows that the generalized active forces  $F_1$  and  $F_2$  for  $S$  in  $A$  can be found by substituting from Eq. (25) into Eqs. (5.1.2), which leads to

$$F_1 = m_1g \cos \theta - k_1q_1 + k_2(q_2 - q_1) \quad (26)$$

and

$$F_2 = m_2g \cos \theta - k_2(q_2 - q_1) \quad (27)$$

in agreement with Eqs. (4.4.16) and (4.4.17), respectively.

When  $\theta(t)$  is a constant, the potential energy  $V$  as given by Eq. (25) satisfies Eq. (5.1.3) and, therefore, Eq. (5.1.4). Conversely, when  $\theta(t)$  is not a constant, then  $V$  does not satisfy Eq. (5.1.3) and it may be verified with the aid of Eqs. (26) and (27) that Eq. (5.1.4) is violated. As will be seen in Sec. 7.2, these facts play a decisive role in connection with the formulation of integrals of the equations of motion of  $S$ .

Suppose that generalized speeds  $u_1$  and  $u_2$  are defined as

$$u_1 \triangleq \mathbf{v}^{P_1} \cdot \mathbf{k} = \dot{q}_1 \cos \theta - (L_1 + q_1)\dot{\theta} \sin \theta \quad (28)$$

$$u_2 \triangleq \mathbf{v}^{P_2} \cdot \mathbf{k} = \dot{q}_2 \cos \theta - (L_1 + L_2 + q_2)\dot{\theta} \sin \theta \quad (29)$$

rather than as in Eqs. (4.4.5). The relevant equations of Sec. 5.1 then are Eqs. (5.1.8) and (5.1.9), with [solve Eqs. (28) and (29) for  $\dot{q}_1$  and  $\dot{q}_2$  and compare the resulting equations with Eqs. (5.1.8)]

$$W_{11} = \sec \theta \quad W_{12} = 0 \quad X_1 = (L_1 + q_1)\dot{\theta} \tan \theta \quad (30)$$

$$W_{21} = 0 \quad W_{22} = \sec \theta \quad X_2 = (L_1 + L_2 + q_2)\dot{\theta} \tan \theta \quad (31)$$

Consequently,  $F_1$  and  $F_2$  are now given by

$$F_1 \underset{(5.1.9)}{=} - \left( \frac{\partial V}{\partial q_1} W_{11} + \frac{\partial V}{\partial q_2} W_{21} \right) \underset{(25,30,31)}{=} m_1 g - [k_1 q_1 - k_2(q_2 - q_1)] \sec \theta \quad (32)$$

$$F_2 \underset{(5.1.9)}{=} - \left( \frac{\partial V}{\partial q_1} W_{12} + \frac{\partial V}{\partial q_2} W_{22} \right) \underset{(25,30,31)}{=} m_2 g - k_2(q_2 - q_1) \sec \theta \quad (33)$$

rather than by Eqs. (26) and (27). Moreover, solving Eqs. (32) and (33) [with the aid of Eqs. (30) and (31)] for  $\partial V/\partial q_1$  and  $\partial V/\partial q_2$  and then using Eq. (5.1.6), one recovers  $V$  as given by Eq. (25), thus verifying that the choice of generalized speeds affects the generalized active forces but not the potential energy of  $S$ .

### 5.3 DISSIPATION FUNCTIONS

If  $S$  is a simple nonholonomic system (see Sec. 2.13) possessing generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10) and generalized speeds  $u_1, \dots, u_n$  (see Sec. 2.12) in a reference frame  $A$ , with  $u_{p+1}, \dots, u_n$  dependent upon  $u_1, \dots, u_p$  in accordance with Eqs. (2.13.1), and  $(\tilde{F}_1)_C, \dots, (\tilde{F}_p)_C$  are the contributions to the generalized active forces  $\tilde{F}_1, \dots, \tilde{F}_p$  (see Sec. 4.4), respectively, of a set  $C$  of contact forces acting on particles of  $S$ , there may exist a function  $\mathcal{F}$  of  $q_1, \dots, q_n, u_1, \dots, u_p$ , and  $t$  such that

$$(\tilde{F}_r)_C = - \frac{\partial \mathcal{F}}{\partial u_r} \quad (r = 1, \dots, p) \quad (1)$$

Under these circumstances,  $\mathcal{F}$  is called a *dissipation function* for  $C$ .

**Example** Referring to the example in Sec. 4.7, note that  $n = 2$  and  $p = n$  (because there are no nonholonomic constraints), and let  $C$  be the set of contact forces exerted on the rods  $A$  and  $B$  by the viscous fluid. Then  $(F_1)_C$  and  $(F_2)_C$ , the contributions of  $C$  to the generalized active forces  $F_1$  and  $F_2$ , respectively, are given by

$$(F_1)_C \underset{(4.7.8)}{=} 0 \quad (F_2)_C \underset{(4.7.9,4.7.3)}{=} -\delta u_2 \quad (2)$$

and  $\mathcal{F}$ , a function of  $q_1, q_2, u_1$ , and  $u_2$  defined as

$$\mathcal{F} \triangleq \frac{1}{2} \delta u_2^2 \quad (3)$$

is a dissipation function for  $C$  because

$$\frac{\partial \mathcal{F}}{\partial u_1} = 0 = -(F_1)_C \quad (4)$$

and

$$\frac{\partial \mathcal{F}}{\partial u_2} = \delta u_2 = -(F_2)_C \quad (5)$$

so that Eqs. (1) are satisfied for  $r = 1, \dots, p$ .

## 5.4 KINETIC ENERGY

The *kinetic energy*  $K$  of a set  $S$  of  $v$  particles  $P_1, \dots, P_v$  in a reference frame  $A$  is defined as

$$K \triangleq \frac{1}{2} \sum_{i=1}^v m_i (\mathbf{v}^{P_i})^2 \quad (1)$$

where  $m_i$  is the mass of  $P_i$  and  $\mathbf{v}^{P_i}$  is the velocity of  $P_i$  in  $A$ .

When a subset of  $S$  forms a rigid body  $B$ , then  $K_B$ , the contribution of  $B$  to  $K$ , can be expressed as

$$K_B = K_\omega + K_v \quad (2)$$

where  $K_\omega$ , called the *rotational* kinetic energy of  $B$  in  $A$ , and  $K_v$ , called the *translational* kinetic energy of  $B$  in  $A$ , depend, respectively, on the angular velocity  $\omega$  of  $B$  in  $A$  and the central inertia dyadic  $\mathbf{I}$  of  $B$ , and on the velocity  $\mathbf{v}$  in  $A$  of the mass center  $B^*$  of  $B$  and the mass  $m$  of  $B$ . Specifically,

$$K_\omega \triangleq \frac{1}{2} \omega \cdot \mathbf{I} \cdot \omega \quad (3)$$

and

$$K_v \triangleq \frac{1}{2} m \mathbf{v}^2 \quad (4)$$

Furthermore, the kinetic energy of rotation of  $B$  in  $A$  is given also by

$$K_\omega = \frac{1}{2} I \omega^2 \quad (5)$$

where  $I$  is the moment of inertia of  $B$  about the line that passes through  $B^*$  and is parallel to  $\omega$  (in general,  $I$  is time-dependent), and by

$$K_\omega = \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \omega_j I_{jk} \omega_k \quad (6)$$

where  $I_{jk}$  ( $j, k = 1, 2, 3$ ) are inertia scalars of  $B$  relative to  $B^*$  (see Sec. 3.3) for any three mutually perpendicular unit vectors (in general,  $I_{jk}$  is time-dependent), and  $\omega_1, \omega_2, \omega_3$  are the associated measure numbers of  $\omega$ . Finally, if  $I_1, I_2, I_3$  are

central principal moments of inertia of  $B$  (see Sec. 3.8), and  $\omega_1, \omega_2, \omega_3$  are the associated measure numbers of  $\omega$ , then

$$K_\omega = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \quad (7)$$

**Derivations** Letting  $\mathbf{r}_i$  be the position vector from  $B^*$  to  $P_i$ , a generic particle of  $B$ , one can write

$$\mathbf{v}^{P_i} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i \quad (2.7.1) \quad (8)$$

where  $\mathbf{v}$  is the velocity of  $B^*$  in  $A$ . Hence,

$$\begin{aligned} K &= \frac{1}{2} \sum_{i=1}^v m_i [\mathbf{v}^2 + 2\mathbf{v} \cdot \boldsymbol{\omega} \times \mathbf{r}_i + (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \frac{1}{2} \left( \sum_{i=1}^v m_i \right) \mathbf{v}^2 + \mathbf{v} \cdot \boldsymbol{\omega} \times \sum_{i=1}^v m_i \mathbf{r}_i + \frac{1}{2} \boldsymbol{\omega} \cdot \sum_{i=1}^v m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \frac{1}{2} m \mathbf{v}^2 + \underset{(3.1.1)}{0} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} = \underset{(4)}{K_v} + \underset{(3)}{K_\omega} \end{aligned} \quad (9)$$

in agreement with Eq. (2).

Let  $\mathbf{n}_\omega$  and  $\omega$  be a unit vector and a scalar, respectively, such that

$$\boldsymbol{\omega} = \mathbf{n}_\omega \omega \quad (10)$$

Then

$$K_\omega = \frac{1}{2} \omega^2 \mathbf{n}_\omega \cdot \mathbf{I} \cdot \mathbf{n}_\omega \quad (3,10) \quad (11)$$

Now,

$$\omega^2 = \boldsymbol{\omega}^2 \quad (10) \quad (12)$$

and

$$\mathbf{n}_\omega \cdot \mathbf{I} \cdot \mathbf{n}_\omega = I \quad (3.5.20) \quad (13)$$

Substitution from Eqs. (12) and (13) into Eq. (11) thus leads to Eq. (5).

When  $\boldsymbol{\omega}$  is expressed as

$$\boldsymbol{\omega} = \sum_{i=1}^3 \omega_i \mathbf{n}_i \quad (14)$$

where  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are any mutually perpendicular unit vectors, then

$$\begin{aligned} K_\omega &= \frac{1}{2} \sum_{i=1}^3 \omega_i \mathbf{n}_i \cdot \sum_{j=1}^3 \sum_{k=1}^3 I_{jk} \mathbf{n}_j \mathbf{n}_k \cdot \sum_{i=1}^3 \omega_i \mathbf{n}_i \\ &= \frac{1}{2} \left( \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \omega_i \mathbf{n}_i \cdot \mathbf{n}_j I_{jk} \mathbf{n}_k \right) \cdot \sum_{i=1}^3 \omega_i \mathbf{n}_i \end{aligned} \quad (3,14) \quad (3.5.22) \quad (15)$$

or, since  $\mathbf{n}_i \cdot \mathbf{n}_j$  vanishes except when  $i = j$ , and is equal to unity when  $i = j$ ,

$$\begin{aligned} K_{\omega} &= \frac{1}{2} \left( \sum_{j=1}^3 \sum_{k=1}^3 \omega_j I_{jk} \mathbf{n}_k \right) \cdot \sum_{i=1}^3 \omega_i \mathbf{n}_i \\ &= \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \sum_{i=1}^3 \omega_j I_{jk} \omega_i \mathbf{n}_k \cdot \mathbf{n}_i \end{aligned} \quad (16)$$

But  $\mathbf{n}_k \cdot \mathbf{n}_i = 0$  except when  $i = k$ , and is equal to unity when  $i = k$ . Hence, Eq. (16) reduces to Eq. (6). Finally, when  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are parallel to central principal axes of  $B$ , then  $I_{jk}$  vanishes except for  $j = k$  [see Eq. (3.8.2)], and Eq. (6) yields

$$K_{\omega} = \frac{1}{2} (I_{11} \omega_1^2 + I_{22} \omega_2^2 + I_{33} \omega_3^2) \quad (17)$$

which establishes the validity of Eq. (7).

**Example** Suppose that the system  $S$  considered in the example in Sec. 4.8 has the following inertia properties: The frame  $A$  has a mass  $m_A$ , and  $A^*$ , the mass center of  $A$ , is situated on line  $DE$ , at a distance  $a$  from  $D$ ; the moment of inertia of  $A$  about a line passing through  $A^*$  and parallel to  $\mathbf{a}_1$  has the value  $I_A$ ; wheels  $B$  and  $C$  are identical, uniform, thin disks of mass  $m_B$  and radius  $R$ . The kinetic energy  $K$  of  $S$  in reference frame  $F$  is to be expressed in terms of the parameters  $a, b, R, m_A, m_B, I_A$ , the generalized coordinates  $q_1, \dots, q_5$  defined in Problem 9.8, and the time-derivatives of these coordinates.

The contribution  $K_A$  of  $A$  to  $K$  is

$$K_A = \frac{1}{2} I_A (\omega^A)^2 + \frac{1}{2} m_A (\mathbf{v}^{A^*})^2 \quad (18)$$

(2)                      (5)                      (4)

with

$$\omega^A = \dot{q}_1 \mathbf{a}_1 \quad (19)$$

and (see Fig. 4.8.2 for  $\mathbf{a}_2, \mathbf{n}_2$ , and  $\mathbf{n}_3$ )

$$\begin{aligned} \mathbf{v}^{A^*} &= \mathbf{v}^D + \omega^A \times (a\mathbf{a}_2) \\ &= (-as_1 \dot{q}_1 + \dot{q}_2) \mathbf{n}_2 + (ac_1 \dot{q}_1 + \dot{q}_3) \mathbf{n}_3 \end{aligned} \quad (20)$$

Thus,

$$K_A = \frac{1}{2} I_A \dot{q}_1^2 + \frac{1}{2} m_A [a^2 \dot{q}_1^2 - 2a \dot{q}_1 (s_1 \dot{q}_2 - c_1 \dot{q}_3) + \dot{q}_2^2 + \dot{q}_3^2] \quad (21)$$

(18-20)

$K_B$ , the contribution of  $B$  to  $K$ , can be expressed as

$$K_B = \frac{1}{2} [I_1^B (\omega_1^B)^2 + I_2^B (\omega_2^B)^2 + I_3^B (\omega_3^B)^2] + \frac{1}{2} m_B (\mathbf{v}^{B^*})^2 \quad (22)$$

(2)                      (7)                      (4)

where

$$I_1^B = I_2^B = \frac{m_B R^2}{4} \quad I_3^B = \frac{m_B R^2}{2} \quad (23)$$

$$\omega_1^B = \dot{q}_1 \quad \omega_2^B = 0 \quad \omega_3^B = \dot{q}_4 \quad (24)$$

and

$$\begin{aligned}\mathbf{v}^{B*} &= \mathbf{v}^D + \boldsymbol{\omega}^A \times (-b\mathbf{a}_3) \\ &= (\dot{q}_2 + b\dot{q}_1c_1)\mathbf{n}_2 + (\dot{q}_3 + b\dot{q}_1s_1)\mathbf{n}_3\end{aligned}\quad (25)$$

Hence,

$$K_B = \frac{m_B}{(22-25)} \frac{1}{2} \left[ \left( \frac{R^2}{4} + b^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \frac{R^2}{2} \dot{q}_4^2 + 2b\dot{q}_1(c_1\dot{q}_2 + s_1\dot{q}_3) \right] \quad (26)$$

Similarly,

$$K_C = \frac{m_B}{2} \left[ \left( \frac{R^2}{4} + b^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \frac{R^2}{2} \dot{q}_5^2 - 2b\dot{q}_1(c_1\dot{q}_2 + s_1\dot{q}_3) \right] \quad (27)$$

Consequently, the desired expression for  $K$  is

$$\begin{aligned}K &= K_A + K_B + K_C \\ &\stackrel{(21,26,27)}{=} \frac{1}{2} \left\{ \left[ I_A + m_A a^2 + 2m_B \left( \frac{R^2}{4} + b^2 \right) \right] \dot{q}_1^2 \right. \\ &\quad \left. + (m_A + 2m_B)(\dot{q}_2^2 + \dot{q}_3^2) + \frac{m_B R^2}{2} (\dot{q}_4^2 + \dot{q}_5^2) \right. \\ &\quad \left. - 2m_A a \dot{q}_1 (s_1 \dot{q}_2 - c_1 \dot{q}_3) \right\}\end{aligned}\quad (28)$$

## 5.5 HOMOGENEOUS KINETIC ENERGY FUNCTIONS

If  $S$  is a simple nonholonomic system (see Sec. 2.13) possessing  $n$  generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10),  $n$  generalized speeds  $u_1, \dots, u_n$  (see Sec. 2.12), and  $p$  degrees of freedom (see Sec. 2.13) in a reference frame  $A$ , and Eqs. (2.13.1) and (2.14.5) apply, then the kinetic energy  $K$  of  $S$  in  $A$  (see Sec. 5.4) can be expressed as

$$K = K_0 + K_1 + K_2 \quad (1)$$

where  $K_i$  is a function of  $q_1, \dots, q_n, u_1, \dots, u_p$ , and the time  $t$ , and is *homogeneous and of degree  $i$*  ( $i = 0, 1, 2$ ) in  $u_1, \dots, u_p$ .

The function  $K_2$  is given by

$$K_2 = \frac{1}{2} \sum_{r=1}^p \sum_{s=1}^p m_{rs} u_r u_s \quad (2)$$

where  $m_{rs}$ , called an *inertia coefficient* of  $S$  in  $A$ , is defined in terms of the masses  $m_1, \dots, m_v$  and partial velocities  $\tilde{\mathbf{v}}_r^{P_i}$  ( $i = 1, \dots, v$ ;  $r = 1, \dots, p$ ) in  $A$  (see Sec. 2.14) of the  $v$  particles  $P_1, \dots, P_v$  forming  $S$  as

$$m_{rs} \triangleq \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_s^{P_i} \quad (r, s = 1, \dots, p) \quad (3)$$

so that

$$m_{rs} = m_{sr} \quad (r, s = 1, \dots, p) \quad (4)$$

Inertia coefficients† figure prominently in the theory of small vibrations of mechanical systems. Generally, the most convenient way to determine inertia coefficients is simply to inspect a kinetic energy expression.

**Derivations** When  $\mathbf{v}^{P_i}$ , the velocity in  $A$  of  $P_i$ , a generic particle of  $S$ , is expressed as

$$\mathbf{v}^{P_i} = \sum_{(2.14.4) \ r=1}^p \tilde{\mathbf{v}}_r^{P_i} u_r + \tilde{\mathbf{v}}_t^{P_i} \quad (i = 1, \dots, v) \quad (5)$$

then Eq. (5.4.1) leads to

$$K = \frac{1}{2} \sum_{i=1}^v \sum_{r=1}^p \sum_{s=1}^p m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_s^{P_i} u_r u_s + \sum_{i=1}^v \sum_{r=1}^p m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_t^{P_i} u_r + \frac{1}{2} \sum_{i=1}^v m_i (\tilde{\mathbf{v}}_t^{P_i})^2 \quad (6)$$

and, if  $K_0$ ,  $K_1$ , and  $K_2$  are defined as

$$K_0 \triangleq \frac{1}{2} \sum_{i=1}^v m_i (\tilde{\mathbf{v}}_t^{P_i})^2 \quad (7)$$

$$K_1 \triangleq \sum_{i=1}^v \sum_{r=1}^p m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_t^{P_i} u_r \quad (8)$$

and

$$K_2 \triangleq \frac{1}{2} \sum_{i=1}^v \sum_{r=1}^p \sum_{s=1}^p m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_s^{P_i} u_r u_s \quad (9)$$

respectively, then Eq. (1) follows directly from Eqs. (7)–(9). Moreover,  $K_0$ ,  $K_1$ , and  $K_2$  can be seen to be homogeneous functions of, respectively, degree 0, 1, and 2 in  $u_1, \dots, u_p$ . Furthermore, since the order in which the summations in Eq. (9) are performed is immaterial, Eqs. (9) and (3) yield Eq. (2).

† Strictly speaking, the quantities defined in Eqs. (3) should be called *nonholonomic* inertia coefficients to distinguish them from the quantities obtained when  $\tilde{\mathbf{v}}_r^{P_i}$  and  $\tilde{\mathbf{v}}_s^{P_i}$  are replaced with  $\mathbf{v}_r^{P_i}$  and  $\mathbf{v}_s^{P_i}$ , respectively, and  $p$  is replaced with  $n$ . The latter quantities would be called *holonomic* inertia coefficients of the nonholonomic system  $S$  in  $A$ . As we shall have no occasion to use them, we leave them undefined. When  $S$  is a holonomic system, the distinction between the two kinds of inertia coefficients disappears in any event.

**Example** When Eqs. (2.12.7) are used to define  $u_1$ ,  $u_2$ , and  $u_3$ , the kinetic energy  $K$  in  $A$  of the system  $S$  formed by the particle  $P_1$  of mass  $m_1$  and the sharp-edged circular disk  $D$  of mass  $m_2$  considered in the example in Sec. 2.13 becomes

$$K = \frac{1}{2}[(m_1 + m_2)u_1^2 + m_1u_2^2] + \frac{1}{2}\omega^2[m_1q_1^2 + m_2(q_1 + Lc_3)^2] \quad (10)$$

$K_0$ , the portion of  $K$  that is of degree zero in  $u_1$  and  $u_2$ , is thus seen to be given by

$$K_0 = \frac{1}{2}\omega^2[m_1q_1^2 + m_2(q_1 + Lc_3)^2] \quad (11)$$

Since  $K$  contains no terms of degree 1 in  $u_1$  and  $u_2$ , the function  $K_1$  is equal to zero. Finally,  $K_2$ , the *quadratic* part of  $K$ , as it is frequently called, is

$$K_2 = \frac{1}{2}[(m_1 + m_2)u_1^2 + m_1u_2^2] \quad (12)$$

For  $p = 2$ , as is the case here, Eqs. (2) and (4) yield

$$K_2 = \frac{1}{2}[m_{11}u_1^2 + (m_{12} + m_{21})u_1u_2 + m_{22}u_2^2] \quad (13)$$

Comparing Eqs. (12) and (13), one finds that the inertia coefficients  $m_{11}$ ,  $m_{12}$ ,  $m_{21}$ , and  $m_{22}$  are

$$m_{11} = m_1 + m_2 \quad m_{12} = m_{21} = 0 \quad m_{22} = m_1 \quad (14)$$

## 5.6 KINETIC ENERGY AND GENERALIZED INERTIA FORCES

If and only if

$$\sum_{i=1}^v m_i \mathbf{v}^{P_i} \cdot \dot{\mathbf{v}}_i^{P_i} = 0 \quad (1)$$

then

$$\dot{K}_2 - \dot{K}_0 = - \sum_{r=1}^p \tilde{F}_r^* u_r \quad (2)$$

where  $\tilde{F}_r^*$  is the  $r$ th nonholonomic generalized inertia force for  $S$  in  $A$  (see Sec. 4.11), and all other symbols have the same meanings as in Sec. 5.5. It is by virtue of these facts that kinetic energy plays an important part in the construction of integrals of equations of motion, as will be shown in Sec. 7.2.

Equation (1) is guaranteed to be satisfied, and Eq. (2) therefore applies, whenever the following  $(n + 1)^2$  equations are satisfied:

$$\frac{\partial K}{\partial t} + \sum_{s=1}^n \frac{\partial K}{\partial q_s} \left( X_s + \sum_{r=p+1}^n W_{sr} B_r \right) = 0 \quad (3)$$

$$\frac{d}{dt} \left( X_s + \sum_{r=p+1}^n W_{sr} B_r \right) = 0 \quad (s = 1, \dots, n) \quad (4)$$

$$\frac{\partial X_s}{\partial t} + \sum_{k=1}^n \frac{\partial W_{sk}}{\partial t} u_k = 0 \quad \frac{\partial X_s}{\partial q_r} + \sum_{k=1}^n \frac{\partial W_{sk}}{\partial q_r} u_k = 0 \quad (r, s = 1, \dots, n) \quad (5)$$



where  $K$  is regarded as a function of  $q_1, \dots, q_n, u_1, \dots, u_p$ , and  $t$ , and,  $X_s, W_{sr}$ , and  $B_r$  have the same meanings as in Eqs. (2.14.5) and (2.13.1).

Equations (3) and (4) reduce to simpler relationships when  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ) and/or when  $S$  is a holonomic system, for, when  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ), then  $X_r = 0$  and  $W_{rs} = \delta_{rs}$  ( $r, s = 1, \dots, n$ ), and, when  $S$  is a holonomic system, then  $B_r = 0$  ( $r = p + 1, \dots, n$ ).

When  $K$  is regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , then  $\tilde{F}_r^*$  can be expressed as

$$\tilde{F}_r^* = - \sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (6)$$

where  $W_{sr}$  and  $A_{kr}$  have the same meaning as in Eqs. (2.14.5) and (2.13.1), respectively. Once again, there are simplifications when  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ) and/or when  $S$  is a holonomic system. When  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ),

$$\tilde{F}_r^* = - \left[ \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} + \sum_{s=p+1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) C_{sr} \right] \quad (r = 1, \dots, p) \quad (7)$$

When  $S$  is a holonomic system, but Eqs. (2.14.5) apply,  $\tilde{F}_r^*$  is replaced with  $F_r^*$  and

$$F_r^* = - \sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) W_{sr} \quad (r = 1, \dots, n) \quad (8)$$

Finally, when  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ) and  $S$  is a holonomic system, then

$$F_r^* = - \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} \right) \quad (r = 1, \dots, n) \quad (9)$$

One can use Eqs. (6)–(9) to find expressions for generalized inertia forces, but frequently it is inefficient to use this approach.

**Derivations** The acceleration  $\mathbf{a}^{P_i}$  of a generic particle  $P_i$  of  $S$  in  $A$ , found by differentiating Eq. (5.5.5) with respect to  $t$  in  $A$ , is

$$\mathbf{a}^{P_i} = \sum_{s=1}^p \left( \dot{\tilde{\mathbf{v}}}_s^{P_i} u_s + \tilde{\mathbf{v}}_s^{P_i} \dot{u}_s \right) + \dot{\tilde{\mathbf{v}}}_t^{P_i} \quad (i = 1, \dots, v) \quad (10)$$

where the dots over  $\tilde{\mathbf{v}}_r^{P_i}$  and  $\tilde{\mathbf{v}}_t^{P_i}$  denote time-differentiation in  $A$ . Hence, with the aid of Eqs. (4.11.1) and (4.11.3), one can express the right-hand member of Eq. (2) as

$$- \sum_{r=1}^p \tilde{F}_r^* u_r = \sum_{r=1}^p \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \left[ \sum_{s=1}^p \left( \dot{\tilde{\mathbf{v}}}_s^{P_i} u_s + \tilde{\mathbf{v}}_s^{P_i} \dot{u}_s \right) + \dot{\tilde{\mathbf{v}}}_t^{P_i} \right] u_r \quad (11)$$

As for the left-hand member, we note that

$$\dot{K}_0 = \sum_{(5.5.7) \ i=1}^v m_i \tilde{\mathbf{v}}_t^{P_i} \cdot \dot{\tilde{\mathbf{v}}}_t^{P_i} \quad (12)$$

and that

$$\begin{aligned} \dot{K}_2 &= \sum_{(5.5.9)}^v \sum_{r=1}^p \sum_{s=1}^p m_i \left( \tilde{\mathbf{v}}_r^{P_i} \cdot \dot{\tilde{\mathbf{v}}}_s^{P_i} u_r u_s + \tilde{\mathbf{v}}_r^{P_i} \cdot \tilde{\mathbf{v}}_s^{P_i} u_r \dot{u}_s \right) \\ &= - \sum_{(11)}^p \tilde{F}_r^* u_r - \sum_{i=1}^v \sum_{r=1}^p m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \dot{\tilde{\mathbf{v}}}_t^{P_i} u_r \end{aligned} \quad (13)$$

so that

$$\dot{K}_2 - \dot{K}_0 = - \sum_{(12.13)}^p \tilde{F}_r^* u_r - \sum_{i=1}^v m_i \sum_{r=1}^p \left( \tilde{\mathbf{v}}_r^{P_i} u_r + \tilde{\mathbf{v}}_t^{P_i} \right) \cdot \dot{\tilde{\mathbf{v}}}_t^{P_i} \quad (14)$$

or, in view of Eqs. (5.5.5),

$$\dot{K}_2 - \dot{K}_0 = - \sum_{(14)}^p \tilde{F}_r^* u_r - \sum_{i=1}^v m_i \mathbf{v}^{P_i} \cdot \dot{\tilde{\mathbf{v}}}_t^{P_i} \quad (15)$$

Hence, when Eq. (1) is satisfied, then so is Eq. (2), and vice-versa. We will now show that Eq. (1) applies whenever Eqs. (3)–(5) are satisfied. To this end, we first express  $\tilde{\mathbf{v}}_t^{P_i}$  as

$$\tilde{\mathbf{v}}_t^{P_i} = \mathbf{v}_t^{P_i} + \sum_{s=p+1}^n \mathbf{v}_s^{P_i} B_s \quad (16)$$

and use Eqs. (2.14.11) and (2.14.12) to write

$$\mathbf{v}_s^{P_i} = \sum_{r=1}^n \frac{\partial \mathbf{p}_i}{\partial q_r} W_{rs} \quad (s = 1, \dots, n) \quad (17)$$

$$\mathbf{v}_t^{P_i} = \sum_{s=1}^n \frac{\partial \mathbf{p}_i}{\partial q_s} X_s + \frac{\partial \mathbf{p}_i}{\partial t} \quad (18)$$

where  $\mathbf{p}_i$  is the position vector from a point fixed in  $A$  to  $P_i$ . Substituting from Eqs. (18) and (17) into Eq. (16), we thus have

$$\begin{aligned} \tilde{\mathbf{v}}_t^{P_i} &= \sum_{s=1}^n \frac{\partial \mathbf{p}_i}{\partial q_s} X_s + \frac{\partial \mathbf{p}_i}{\partial t} + \sum_{s=p+1}^n \sum_{r=1}^n \frac{\partial \mathbf{p}_i}{\partial q_r} W_{rs} B_s \\ &= \sum_{s=1}^n \left[ \frac{\partial \mathbf{p}_i}{\partial q_s} \left( \sum_{r=p+1}^n W_{sr} B_r + X_s \right) \right] + \frac{\partial \mathbf{p}_i}{\partial t} \end{aligned} \quad (19)$$

and, after time-differentiation in  $A$ ,

$$\dot{\tilde{\mathbf{v}}}_t^{P_i} = \sum_{(19.4)}^n \left[ \frac{d}{dt} \left( \frac{\partial \mathbf{p}_i}{\partial q_s} \right) \left( \sum_{r=p+1}^n W_{sr} B_r + X_s \right) \right] + \frac{d}{dt} \left( \frac{\partial \mathbf{p}_i}{\partial t} \right) \quad (20)$$

Now, since  $\mathbf{p}_i$  is here regarded as a function of  $q_1, \dots, q_n$ , and  $t$ ,

$$\frac{d}{dt} \left( \frac{\partial \mathbf{p}_i}{\partial q_s} \right) = \sum_{(1.9.1)}^n \frac{\partial^2 \mathbf{p}_i}{\partial q_r \partial q_s} \left( \sum_{k=1}^n W_{rk} u_k + X_r \right) + \frac{\partial^2 \mathbf{p}_i}{\partial t \partial q_s} \quad (s = 1, \dots, n) \quad (21)$$

(2.14.5)

and

$$\frac{d}{dt} \left( \frac{\partial \mathbf{p}_i}{\partial t} \right) \underset{(1.9.1)}{=} \sum_{r=1}^n \frac{\partial^2 \mathbf{p}_i}{\partial t \partial q_r} \left( \sum_{k=1}^n W_{rk} u_k + X_r \right) \underset{(2.14.5)}{+} \frac{\partial^2 \mathbf{p}_i}{\partial t^2} \quad (22)$$

Moreover, with  $\dot{\mathbf{p}}_i$  in place of  $\mathbf{v}^{P_i}$ , it follows from Eqs. (5.4.1), (1.9.1), and (2.14.5) that, when Eqs. (5) are satisfied,

$$\frac{\partial K}{\partial q_s} = \sum_{i=1}^v m_i \dot{\mathbf{p}}_i \cdot \left\{ \sum_{r=1}^n \left[ \frac{\partial^2 \mathbf{p}_i}{\partial q_s \partial q_r} \left( \sum_{k=1}^n W_{rk} u_k + X_r \right) \right] + \frac{\partial^2 \mathbf{p}_i}{\partial q_s \partial t} \right\} \quad (s = 1, \dots, n) \quad (23)$$

and

$$\frac{\partial K}{\partial t} = \sum_{i=1}^v m_i \dot{\mathbf{p}}_i \cdot \left\{ \sum_{r=1}^n \left[ \frac{\partial^2 \mathbf{p}_i}{\partial t \partial q_r} \left( \sum_{k=1}^n W_{rk} u_k + X_r \right) \right] + \frac{\partial^2 \mathbf{p}_i}{\partial t^2} \right\} \quad (24)$$

Consequently,

$$\sum_{i=1}^v m_i \mathbf{v}^{P_i} \cdot \dot{\mathbf{v}}^{P_i} \underset{(20-24)}{=} \sum_{s=1}^n \frac{\partial K}{\partial q_s} \left( \sum_{r=p+1}^n W_{sr} B_r + X_s \right) + \frac{\partial K}{\partial t} \underset{(3)}{=} 0 \quad (25)$$

The validity of Eqs. (6) is established by writing

$$\begin{aligned} \tilde{F}_r^* & \underset{(4.11.1, 4.11.3)}{=} - \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{a}_i \\ & \underset{(2.15.8)}{=} - \sum_{i=1}^v m_i \frac{1}{2} \sum_{s=1}^n \left\{ \left[ \frac{d}{dt} \frac{\partial (\mathbf{v}^{P_i})^2}{\partial \dot{q}_s} - \frac{\partial (\mathbf{v}^{P_i})^2}{\partial q_s} \right] \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \right\} \\ & = - \sum_{s=1}^n \left\langle \frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_s} \left[ \frac{1}{2} \sum_{i=1}^v m_i (\mathbf{v}^{P_i})^2 \right] \right\} \right. \\ & \quad \left. - \frac{\partial}{\partial q_s} \left[ \frac{1}{2} \sum_{i=1}^v m_i (\mathbf{v}^{P_i})^2 \right] \right\rangle \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \\ & \underset{(5.4.1)}{=} - \sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) \quad (r = 1, \dots, p) \quad (26) \end{aligned}$$

Finally, Eqs. (7)–(9) are special cases of Eqs. (6).

**Example** For the example in Sec. 5.5,  $n$  and  $p$  have the values 3 and 2, respectively; Eqs. (5.1.34) and (5.1.35) show that  $X_s$  and  $B_r$  vanish for  $s = 1, \dots, n$  and  $r = p + 1, \dots, n$ ; and  $W_{sr}$  ( $s, r = 1, 2, 3$ ) can be found by inspection of Eqs. (2.12.8) and (2.14.5). Equations (3) and (4) are thus seen to be satisfied and Eq. (2) therefore applies if  $\partial K / \partial t = 0$ , since this is all Eq. (3) now requires. That this requirement is, in fact, fulfilled can be seen by inspecting Eq. (5.5.10)

when one keeps in mind that here  $K$  is to be regarded as a function of  $q_1, q_2, q_3, u_1, u_2$ , and  $t$ , and that  $\omega$  is a constant.

To verify that Eq. (2) is satisfied, we first evaluate  $\dot{K}_2 - \dot{K}_0$  by reference to Eqs. (5.5.11) and (5.5.12), obtaining

$$\begin{aligned}\dot{K}_2 - \dot{K}_0 &= (m_1 + m_2)u_1\dot{u}_1 + m_1u_2\dot{u}_2 \\ &\quad - \omega^2[m_1q_1\dot{q}_1 + m_2(q_1 + Lc_3)(\dot{q}_1 - Ls_3\dot{q}_3)] \\ &= \{(m_1 + m_2)\dot{u}_1 - \omega^2[m_1q_1 + m_2(q_1 + Lc_3)]c_3\}u_1 \\ &\quad + m_1(\dot{u}_2 + \omega^2q_1s_3)u_2\end{aligned}\quad (27)$$

and then, to illustrate the use of Eqs. (6), form  $\tilde{F}_1^*$  as

$$\begin{aligned}\tilde{F}_1^* &= -\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_1} - \frac{\partial K}{\partial q_1}\right)(W_{11} + W_{13}A_{31}) \\ &\quad -\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_2} - \frac{\partial K}{\partial q_2}\right)(W_{21} + W_{23}A_{31}) \\ &\quad -\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_3} - \frac{\partial K}{\partial q_3}\right)(W_{31} + W_{33}A_{31}) \\ &= -\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_1} - \frac{\partial K}{\partial q_1}\right)c_3 - \left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_2} - \frac{\partial K}{\partial q_2}\right)s_3\end{aligned}\quad (28)$$

Before performing the differentiations indicated in Eq. (28), one must express  $K$  as a function of  $q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3$ , and  $t$ , leaving the motion constraint equation  $u_3 = -u_2/L$  out of account. The required expression for  $K$ , available in Problem 10.1, is

$$\begin{aligned}K &= \frac{1}{2}\omega^2[m_1q_1^2 + m_2(q_1 + Lc_3)^2] + \frac{1}{2}(m_1 + m_2)(\dot{q}_1^2 + \dot{q}_2^2) \\ &\quad - m_2L\left(\dot{q}_1s_3 - \dot{q}_2c_3 - \frac{L}{2}\dot{q}_3\right)\dot{q}_3\end{aligned}\quad (29)$$

Hence,

$$\frac{\partial K}{\partial \dot{q}_1} = (m_1 + m_2)\dot{q}_1 - m_2Ls_3\dot{q}_3\quad (30)$$

$$\frac{\partial K}{\partial \dot{q}_2} = (m_1 + m_2)\dot{q}_2 + m_2Lc_3\dot{q}_3\quad (31)$$

$$\frac{\partial K}{\partial q_1} = \omega^2[m_1q_1 + m_2(q_1 + Lc_3)]\quad (32)$$

$$\frac{\partial K}{\partial q_2} = 0\quad (33)$$

and substitution into Eq. (28) produces

$$\begin{aligned}
 \tilde{F}_1^* &= -\{(m_1 + m_2)\ddot{q}_1 - m_2 L(c_3 \dot{q}_3^2 + s_3 \ddot{q}_3) \\
 &\quad - \omega^2[m_1 q_1 + m_2(q_1 + Lc_3)]\}c_3 \\
 &\quad - [(m_1 + m_2)\ddot{q}_2 - m_2 L(s_3 \dot{q}_3^2 - c_3 \ddot{q}_3)]s_3 \\
 &= -(m_1 + m_2)(\ddot{q}_1 c_3 + \ddot{q}_2 s_3) + m_2 L \dot{q}_3^2 \\
 &\quad + \omega^2[m_1 q_1 + m_2(q_1 + Lc_3)]c_3
 \end{aligned} \tag{34}$$

from which we must eliminate the time-derivatives of  $q_1$ ,  $q_2$ , and  $q_3$  to accomplish our ultimate objective. This can be done by using Eqs. (2.12.8), which permits one to write

$$\begin{aligned}
 \ddot{q}_1 c_3 + \ddot{q}_2 s_3 &= (\dot{u}_1 c_3 - u_1 \dot{q}_3 s_3 - \dot{u}_2 s_3 - u_2 \dot{q}_3 c_3)c_3 \\
 &\quad + (\dot{u}_1 s_3 + u_1 \dot{q}_3 c_3 + \dot{u}_2 c_3 - u_2 \dot{q}_3 s_3)s_3 \\
 &= \dot{u}_1 - u_2 u_3
 \end{aligned} \tag{35}$$

and  $\tilde{F}_1^*$  then is seen to be given by

$$\tilde{F}_1^* = -(m_1 + m_2)(\dot{u}_1 - u_2 u_3) + m_2 L u_3^2 + \omega^2[m_1 q_1 + m_2(q_1 + Lc_3)]c_3 \tag{36}$$

Finally, after eliminating  $u_3$  with the aid of Eq. (2.13.12), we arrive at

$$\tilde{F}_1^* = -(m_1 + m_2)\dot{u}_1 - m_1 \frac{u_2^2}{L} + \omega^2[m_1 q_1 + m_2(q_1 + Lc_3)]c_3 \tag{37}$$

(in agreement with the results in Problem 8.15).

Using the expression for  $\tilde{F}_2^*$  available in the results in Problem 8.15, we can write the right-hand member of Eq. (2) as

$$\begin{aligned}
 -(\tilde{F}_1^* u_1 + \tilde{F}_2^* u_2) &= \left\{ (m_1 + m_2)\dot{u}_1 + m_1 \frac{u_2^2}{L} \right. \\
 &\quad \left. - \omega^2[m_1 q_1 + m_2(q_1 + Lc_3)]c_3 \right\} u_1 \\
 &\quad + m_1 \left( \dot{u}_2 - \frac{u_1 u_2}{L} + \omega^2 q_1 s_3 \right) u_2
 \end{aligned} \tag{38}$$

which reduces to the right-hand member of Eq. (27) in conformity with Eq. (2).

## FORMULATION OF EQUATIONS OF MOTION

In Sec. 6.1, the notion of a Newtonian reference frame is presented, and it is shown that dynamical differential equations can be formulated easily, once expressions for generalized active forces and generalized inertia forces are in hand. Questions regarding reference frame choices are examined in Sec. 6.2, and Sec. 6.3 deals with the formulation of equations intended for the determination of forces and/or torques that do not come into evidence explicitly in equations of motion unless special measures are taken. Section 6.4 contains a detailed exposition of a method for generating linearized forms of dynamical equations. The remainder of the chapter is devoted to consideration of three kinds of motion that deserve attention because of their practical importance, namely rest, steady motion, and motions resembling states of rest, which are treated in Secs. 6.5, 6.6, and 6.7, respectively.

### 6.1 DYNAMICAL EQUATIONS

There exist reference frames  $N$  such that, if  $S$  is a simple nonholonomic system possessing  $p$  degrees of freedom in  $N$  (see Sec. 2.13), and  $\tilde{F}_r$  and  $\tilde{F}_r^*$  ( $r = 1, \dots, p$ ) are, respectively, the nonholonomic generalized active forces (see Sec. 4.4) and the nonholonomic generalized inertia forces (see Sec. 4.11) for  $S$  in  $N$ , then the equations

$$\tilde{F}_r + \tilde{F}_r^* = 0 \quad (r = 1, \dots, p) \quad (1)$$

govern all motions of  $S$  in *any* reference frame. The reference frames  $N$  are called “Newtonian” or “inertial” reference frames, and Eqs. (1) are known as *Kane’s dynamical equations*. If  $S$  is a holonomic system, Eqs. (1) are replaced with

$$F_r + F_r^* = 0 \quad (r = 1, \dots, n) \quad (2)$$

(see Secs. 4.4 and 4.11 for  $F_r$  and  $F_r^*$ , respectively), where  $n$  is the number of generalized coordinates of  $S$  in  $N$ .

Ultimately, the justification for regarding a particular reference frame as Newtonian can come only from experiments. One such experiment, first performed by Foucault in 1851, is discussed in the example that follows the derivation of Eqs. (1).

**Derivation** If  $\mathbf{R}_i$  is the resultant of all contact forces and distance forces acting on a typical particle  $P_i$  of  $S$ , and  $\mathbf{a}_i$  is the acceleration of  $P_i$  in a Newtonian reference frame  $N$ , then, in accordance with Newton’s second law,

$$\mathbf{R}_i - m_i \mathbf{a}_i = 0 \quad (i = 1, \dots, v) \quad (3)$$

where  $m_i$  is the mass of  $P_i$  and  $v$  is the number of particles of  $S$ . Dot-multiplication of Eqs. (3) with the partial velocities  $\tilde{\mathbf{v}}_r^{P_i}$  of  $P_i$  in  $N$  (see Sec. 2.14) and subsequent summation yields

$$\sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i + \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot (-m_i \mathbf{a}_i) = 0 \quad (r = 1, \dots, p) \quad (4)$$

(3)

In accordance with Eq. (4.4.1), the first sum in this equation is  $\tilde{F}_r$ ; the second sum is  $\tilde{F}_r^*$ , as may be seen by reference to Eqs. (4.11.1) and (4.11.3). Thus, Eqs. (4) lead directly to Eqs. (1).

**Example** In Fig. 6.1.1,  $E$  represents the Earth, modeled as a sphere centered at a point  $E^*$ , and  $P$  is a particle of mass  $m$ , suspended by means of a light, inextensible string of length  $L$  from a point  $Q$  that is fixed relative to  $E$ . Point  $O$  is the intersection of line  $E^*Q$  with the surface of  $E$ ;  $h$  is the distance from  $O$  to  $Q$ ;  $\mathbf{k}$  is a unit vector parallel to line  $K$ , the Earth’s polar axis;  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are unit vectors pointing southward, eastward, and upward at  $O$ , respectively. Finally,  $F$  represents a reference frame in which  $E^*$  is fixed and in which  $E$  has a simple angular velocity (see Sec. 2.2)  ${}^F\boldsymbol{\omega}^E$  given by

$${}^F\boldsymbol{\omega}^E = \omega \mathbf{k} \quad (5)$$

where  $\omega = 7.29 \times 10^{-5}$  rad/s, so that  $E$  performs in  $F$  one rotation per sidereal day (24 h of sidereal time or 23 h 56 min 4.09054 s of mean solar time). ( $F$  differs from the so-called astronomical reference frame primarily in that one point of  $F$  coincides permanently with a point of  $E$ , the point  $E^*$ . No point of  $E$  is fixed in the astronomical reference frame.)

To bring  $P$  into a general position, one can proceed as follows: Place  $P$  on line  $OQ$ , subject line  $QP$  to a rotation characterized by the vector  $q_1 \mathbf{e}_1$  rad,





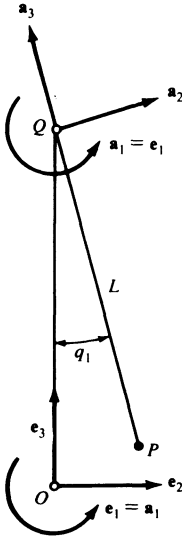


Figure 6.1.2

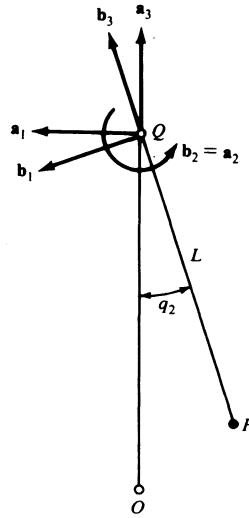


Figure 6.1.3

then, by hypothesis, we have

$${}^F\mathbf{v}^{\bar{E}} = {}^F\boldsymbol{\omega}^E \times \mathbf{p} = \omega \mathbf{k} \times \mathbf{p} \quad (10)$$

(8,9) (5)

As for  ${}^E\mathbf{v}^P$ , it follows directly from Eqs. (6) together with the fact that  ${}^E\mathbf{v}^P \cdot \mathbf{b}_3 = 0$  (because  ${}^E\mathbf{v}^P$  must be perpendicular to  $\mathbf{b}_3$ ) that (see Sec. 1.3)

$${}^E\mathbf{v}^P = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 \quad (11)$$

Thus,

$${}^F\mathbf{v}^P = \omega \mathbf{k} \times \mathbf{p} + u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 \quad (12)$$

(7) (10) (11)

and the partial velocities of  $P$  in  $F$  are simply

$${}^F\mathbf{v}_1^P = \mathbf{b}_1 \quad {}^F\mathbf{v}_2^P = \mathbf{b}_2 \quad (13)$$

(12) (12)

The acceleration of  $P$  in  $F$ , required in connection with the generalized inertia forces for  $P$  in  $F$ , is given by

$${}^F\mathbf{a}^P = {}^F\mathbf{a}^{\bar{E}} + {}^E\mathbf{a}^P + 2{}^F\boldsymbol{\omega}^E \times {}^E\mathbf{v}^P \quad (14)$$

(2.8.2)

with

$$\begin{aligned} {}^F\mathbf{a}^{\bar{E}} &= {}^F\mathbf{a}^{E*} + {}^F\boldsymbol{\omega}^E \times ({}^F\boldsymbol{\omega}^E \times \mathbf{p}) + {}^F\boldsymbol{\alpha}^E \times \mathbf{p} \\ &= 0 + \omega^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{p}) + 0 \end{aligned} \quad (15)$$

(5) (5)

$${}^E\mathbf{a}^P = \frac{d}{dt}({}^E\mathbf{v}^P) = \dot{u}_1 \mathbf{b}_1 + \dot{u}_2 \mathbf{b}_2 \quad (16)$$

(2.6.2) (11)

and

$${}^F\omega^E \times {}^E\mathbf{v}^P = 2\omega\mathbf{k} \times (u_1\mathbf{b}_1 + u_2\mathbf{b}_2) \quad (17)$$

(5) (11)

so that

$${}^F\mathbf{a}^P = \omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{p}) + \dot{u}_1\mathbf{b}_1 + \dot{u}_2\mathbf{b}_2 + 2\omega\mathbf{k} \times (u_1\mathbf{b}_1 + u_2\mathbf{b}_2) \quad (18)$$

(14) (15) (16) (17)

Hence, the generalized inertia forces for  $P$  in  $F$ , given by

$$F_r^* = {}^F\mathbf{v}_r^P \cdot (-m^F\mathbf{a}^P) \quad (r = 1, 2) \quad (19)$$

(4.11.2) (4.11.3)

where  $m$  is the mass of  $P$ , are

$$F_1^* = -m[\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{p}) \cdot \mathbf{b}_1 + \dot{u}_1 - 2\omega u_2\mathbf{k} \cdot \mathbf{b}_3] \quad (20)$$

(19,13,18)

and

$$F_2^* = -m[\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{p}) \cdot \mathbf{b}_2 + \dot{u}_2 + 2\omega u_1\mathbf{k} \cdot \mathbf{b}_3] \quad (21)$$

(19,13,18)

The only contact force acting on  $P$  is the force exerted on  $P$  by the string connecting  $P$  to  $Q$ . Since the string is presumed to be "light," the line of action of this force is regarded as parallel to  $\mathbf{b}_3$ , and the force contributes nothing to the generalized active forces  $F_1$  and  $F_2$  for  $S$  in  $F$  because the partial velocities  ${}^F\mathbf{v}_1^P$  and  ${}^F\mathbf{v}_2^P$  [see Eqs. (13)] are perpendicular to  $\mathbf{b}_3$ . By way of contrast, the gravitational force  $\mathbf{G}$  exerted on  $P$  by  $E$ , given by

$$\mathbf{G} = -mge_3 \quad (22)$$

(4.8.1)

where  $g$  is the gravitational acceleration at  $O$ , does contribute to  $F_1$  and  $F_2$ . Hence,

$$F_r = \mathbf{G} \cdot {}^F\mathbf{v}_r^P = -mge_3 \cdot {}^F\mathbf{v}_r^P \quad (r = 1, 2) \quad (23)$$

(22)

so that

$$F_1 = -mge_3 \cdot \mathbf{b}_1 \quad F_2 = -mge_3 \cdot \mathbf{b}_2 \quad (24)$$

and substitution from Eqs. (24), (20), and (21) into Eqs. (2) yields

$$\dot{u}_1 = -\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{p}) \cdot \mathbf{b}_1 + 2\omega u_2\mathbf{k} \cdot \mathbf{b}_3 - ge_3 \cdot \mathbf{b}_1 \quad (25)$$

$$\dot{u}_2 = -\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{p}) \cdot \mathbf{b}_2 - 2\omega u_1\mathbf{k} \cdot \mathbf{b}_3 - ge_3 \cdot \mathbf{b}_2 \quad (26)$$

The leading term in each of these equations may be dropped because it is approximately four orders of magnitude smaller than the last term in each equation. As for the dot products  $e_3 \cdot \mathbf{b}_1$  and  $e_3 \cdot \mathbf{b}_2$ , reference to Figs. 6.1.2 and 6.1.3 permits one to express these as

$$e_3 \cdot \mathbf{b}_1 = -c_1s_2 \quad e_3 \cdot \mathbf{b}_2 = s_1 \quad (27)$$

and, if  $\phi$  is the angle between line  $K$  and line  $F^*O$  (see Fig. 6.1.1), then  $\mathbf{k} \cdot \mathbf{b}_3$  is given by

$$\mathbf{k} \cdot \mathbf{b}_3 = c_1 c_2 c\phi - s_2 s\phi \quad (28)$$

Hence, if  $F$  is a Newtonian reference frame, the dynamical equations governing all motions of  $P$  are

$$\dot{u}_1 = 2\omega u_2 (c_1 c_2 c\phi - s_2 s\phi) + g c_1 s_2 \quad (29)$$

(2.5) (2.8) (2.7)

$$\dot{u}_2 = -2\omega u_1 (c_1 c_2 c\phi - s_2 s\phi) - g s_1 \quad (30)$$

(2.6) (2.8) (2.7)

Since Eqs. (29) and (30) involve four unknown functions of time, namely,  $u_1$ ,  $u_2$ ,  $q_1$ , and  $q_2$ , they must be supplemented by two additional equations before they can be used to produce motion predictions. The required additional equations are kinematical relationships obtained by taking advantage of the fact that  ${}^E\mathbf{v}^P$  is given not only by Eq. (11), but also by (see Fig. 6.1.3)

$${}^E\mathbf{v}^P = \frac{{}^E d}{dt}(-L\mathbf{b}_3) = -L{}^E\boldsymbol{\omega}^B \times \mathbf{b}_3 \quad (31)$$

(2.6.1) (2.1.2)

where  ${}^E\boldsymbol{\omega}^B$  is the angular velocity in  $E$  of a reference frame  $B$  in which  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , and  $\mathbf{b}_3$  are fixed; that is (see Figs. 6.1.2 and 6.1.3),

$${}^E\boldsymbol{\omega}^B = \dot{q}_1 \mathbf{a}_1 + \dot{q}_2 \mathbf{b}_2 = \dot{q}_1 (c_2 \mathbf{b}_1 + s_2 \mathbf{b}_3) + \dot{q}_2 \mathbf{b}_2 \quad (32)$$

Consequently,

$${}^E\mathbf{v}^P = -L(\dot{q}_2 \mathbf{b}_1 - \dot{q}_1 c_2 \mathbf{b}_2) \quad (33)$$

(31.3.2)

and, comparing this equation with Eq. (11), one concludes that the needed equations are

$$\dot{q}_1 = \frac{u_2}{L c_2} \quad \dot{q}_2 = -\frac{u_1}{L} \quad (34)$$

Since Eqs. (29), (30), and (34) form a set of coupled, *nonlinear* differential equations, they cannot be solved in closed form. However, particular solutions, corresponding to specific initial conditions, can be obtained by solving the equations numerically (a subject discussed in its own right in Sec. 7.5). Suppose, for instance, that  $L = 10$  m,  $\phi = 45^\circ$  and  $q_1$ ,  $q_2$ ,  $u_1$ ,  $u_2$  have the following initial values:

$$q_1(0) = 10^\circ \quad q_2(0) = u_1(0) = u_2(0) = 0 \quad (35)$$

In other words,  $P$  is initially displaced toward the east in such a way that the string makes an angle of  $10^\circ$  with the local vertical, and  $P$  is then released

Table 6.1.1

1 $t$ (s)	$F$ regarded as Newtonian		$E$ regarded as Newtonian	
	2 $q_1$ (deg)	3 $q_2$ (deg)	4 $q_1$ (deg)	5 $q_2$ (deg)
0.0	10.00000	0.00000	10.00000	0.00000
2.0	-3.95462	0.00088	-3.95462	0.00000
4.0	-6.87785	0.00105	-6.87785	0.00000
6.0	9.38832	-0.00306	9.38832	0.00000
8.0	-0.54571	0.00068	-0.54571	0.00000
10.0	-8.95764	0.00442	-8.95764	0.00000
12.0	7.62749	-0.00496	7.62749	0.00000
14.0	2.93028	-0.00180	2.93028	0.00000
16.0	-9.94059	0.00815	-9.94059	0.00000
18.0	4.93175	-0.00475	4.93176	0.00000
20.0	6.04627	-0.00612	6.04628	0.00000
22.0	-9.70738	0.01100	-9.70739	0.00000
24.0	1.63058	-0.00198	1.63058	0.00000
26.0	8.42042	-0.01134	8.42042	0.00000
28.0	-8.28631	0.01180	-8.28632	0.00000
30.0	-1.87098	0.00324	-1.87098	0.00000
32.0	9.76306	-0.01617	9.76307	0.00000
34.0	-5.85006	0.00975	-5.85006	0.00000
36.0	-5.14262	0.01016	-5.14263	0.00000
38.0	9.91107	-0.01925	9.91109	0.00000
40.0	2.69598	0.00456	-2.69599	0.00000
42.0	-7.78299	0.01757	-7.78301	0.00000
44.0	8.84651	-0.01939	8.84653	0.00000
46.0	0.78933	-0.00340	0.78934	0.00000
48.0	-9.46949	0.02393	-9.46952	0.00000
50.0	6.69863	-0.01583	6.69865	0.00000
52.0	4.17764	-0.01310	4.17766	0.00000
54.0	-9.99698	0.02771	-9.99702	0.00000
56.0	3.72919	-0.00841	3.72920	0.00000
58.0	7.05291	-0.02300	7.05295	0.00000
60.0	9.30148	0.02762	-9.30152	0.00000

from a state of rest in  $E$  [see Eqs. (11) and (35)]. The numerical solution of Eqs. (29), (30), and (34) corresponding to these initial conditions (and  $\omega = 7.29 \times 10^{-5}$  rad/s,  $g = 9.81$  m/s<sup>2</sup>) leads to  $q_1$  and  $q_2$  values such as those recorded in columns 2 and 3 of Table 6.1.1. Columns 4 and 5 show the values one obtains when one assumes that  $E$  rather than  $F$  is a Newtonian reference frame, results that can be generated by using Eqs. (29) and (30) with  $\omega = 0$ . The two sets of results can be seen to agree with each other rather well as regards  $q_1$ , but differ from each other markedly as regards  $q_2$ . To determine whether column 3 or column 5 corresponds more nearly to reality, it is helpful to proceed as follows.

Let  $\mathbf{r}$  be the position vector from  $O$  to  $P^*$ , the orthogonal projection of  $P$  on the horizontal plane passing through  $O$ , as shown in Fig. 6.1.4. Then

$$\mathbf{r} = L\mathbf{e}_3 \times (\mathbf{e}_3 \times \mathbf{b}_3) \quad (36)$$

or, since  $\mathbf{b}_3 = s_2\mathbf{e}_1 - s_1c_2\mathbf{e}_2 + c_1c_2\mathbf{e}_3$ ,

$$\mathbf{r} = L(-s_2\mathbf{e}_1 + s_1c_2\mathbf{e}_2) \quad (37)$$

Hence, if  $E_1$  and  $E_2$  are Cartesian coordinate axes passing through  $O$  and parallel to  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , respectively, then the  $E_1$  and  $E_2$  coordinates of  $P^*$  are  $-Ls_2$  and  $Ls_1c_2$ , respectively, and one can plot the path traced out by  $P^*$  in the  $E_1$ - $E_2$  plane, as has been done in Fig. 6.1.5 with data corresponding to columns 2 and 3 of Table 6.1.1; a portion of the  $E_2$  axis represents columns 4 and 5 since, with  $q_2 = 0$ , Eq. (37) reduces to

$$\mathbf{r} = Ls_1\mathbf{e}_2 \quad (38)$$

Figure 6.1.5 shows that the assumption that  $F$  is a Newtonian reference frame leads to the prediction that  $P^*$  must trace out a complicated curve in the  $E_1$ - $E_2$  plane. If it is assumed that  $E$  is a Newtonian reference frame, then the path predicted for  $P^*$  is simply a straight line. Foucault's experiments, performed at the Panthéon in Paris and subsequently duplicated in numerous other locations, revealed that  $P^*$  moves on a curve such as the one in Fig. 6.1.5. Moreover, good quantitative agreement was obtained between experimental data and values predicted mathematically. Hence, regarding  $F$  as a Newtonian reference frame is more realistic than assuming that  $E$  is a Newtonian reference frame. But this does not mean that  $F$  is, in fact, a Newtonian reference frame or that one may not regard  $E$  as such a reference frame in certain contexts.

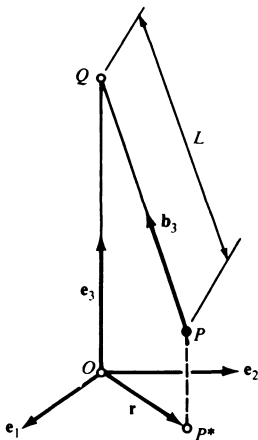


Figure 6.1.4

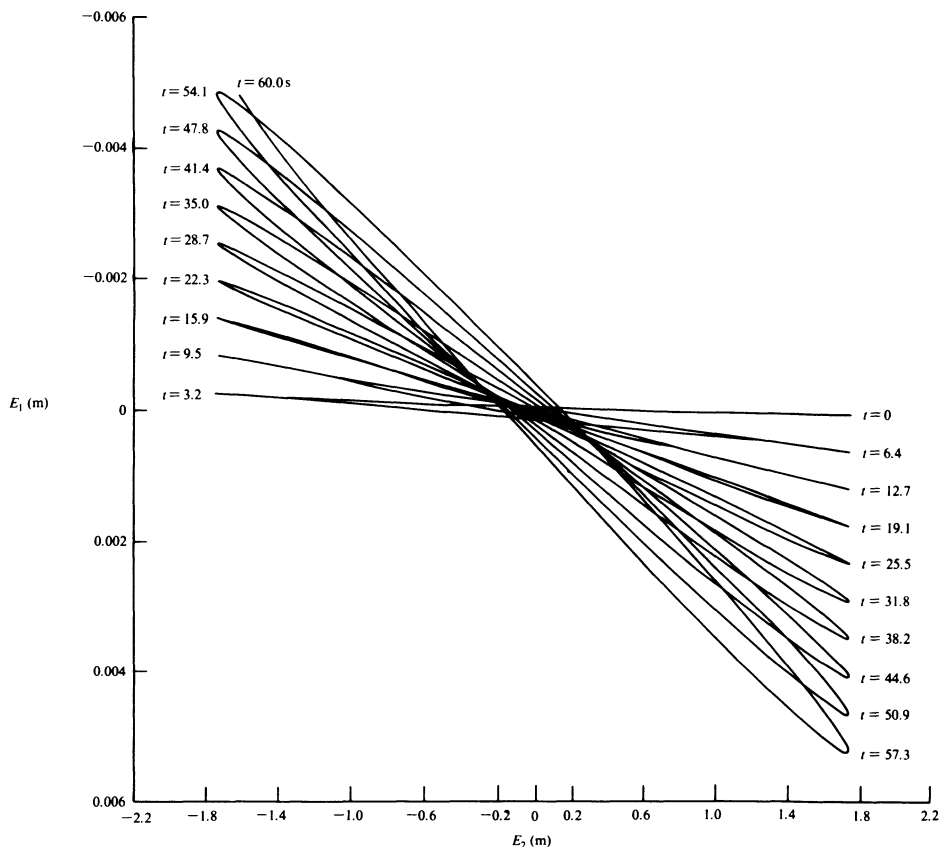


Figure 6.1.5

## 6.2 SECONDARY NEWTONIAN REFERENCE FRAMES

Astronomical observations furnish a large body of data showing that the reference frame  $F$  of the example in Sec. 6.1 is not a Newtonian reference frame, because motion predictions based on Eqs. (6.1.1) or (6.1.2) together with the assumption that  $F$  is such a reference frame conflict with the data. This fact suggests the following question: Why does the hypothesis that  $F$  is a Newtonian reference frame lead to a satisfactory description of the motion of a pendulum relative to the Earth, but to an incorrect description of, for example, the motion of the Moon relative to the Earth? This question and a number of related ones can be answered in the light of the following theorem:

If  $N'$  is a reference frame performing a prescribed motion relative to a Newtonian reference frame  $N$ , then  $N'$  is a Newtonian reference frame throughout some time interval if and only if throughout this time interval the acceleration in  $N$  of every point of  $N'$  is equal to zero. When a reference frame moves in such a way that

the acceleration of each of its points in a Newtonian reference frame is equal to zero, it is called a *secondary Newtonian reference frame*.

**Proof** If  $P$  is any particle of a system  $S$ , then the partial velocities of  $P$  in  $N$  and in  $N'$  are equal to each other since the velocities of  $P$  in  $N$  and in  $N'$  differ from each other only due to the motion of  $N'$  relative to  $N$ , which, being prescribed, does not involve the generalized speeds for  $S$  in  $N$ , and hence does not affect the partial velocities. Consequently, the generalized active force  $\tilde{F}_r$  for  $S$  in  $N$  is necessarily equal to the generalized active force  $\tilde{F}'_r$  ( $r = 1, \dots, p$ ) for  $S$  in  $N'$ , no matter how  $N'$  moves in  $N$ ; but the generalized inertia force  $\tilde{F}_r^*$  for  $S$  in  $N$  can differ from the generalized inertia force  $\tilde{F}'_r^*$  for  $S$  in  $N'$  due to differences in the accelerations  ${}^N\mathbf{a}^P$  and  ${}^{N'}\mathbf{a}^P$ . Now,

$${}^N\mathbf{a}^P = {}^{N'}\mathbf{a}^P + {}^N\mathbf{a}^{\bar{N}'} + 2{}^N\boldsymbol{\omega}^{N'} \times {}^{N'}\mathbf{v}^P \quad (1)$$

(2.8.1)

where  $\bar{N}'$  is the point of  $N'$  that coincides with  $P$ . Furthermore, if  $O$  is any point fixed in  $N'$ , then

$${}^N\mathbf{a}^{\bar{N}'} = {}^N\mathbf{a}^O + {}^N\boldsymbol{\alpha}^{N'} \times \mathbf{r} + {}^N\boldsymbol{\omega}^{N'} \times ({}^N\boldsymbol{\omega}^{N'} \times \mathbf{r}) \quad (2)$$

(2.7.1)

where  $\mathbf{r}$  is the position vector from  $O$  to  $\bar{N}'$ . Hence, if the acceleration in  $N$  of every point of  $N'$  is equal to zero, so that

$${}^N\mathbf{a}^{\bar{N}'} = {}^N\mathbf{a}^O = 0 \quad (3)$$

then

$${}^N\boldsymbol{\alpha}^{N'} \times \mathbf{r} + {}^N\boldsymbol{\omega}^{N'} \times ({}^N\boldsymbol{\omega}^{N'} \times \mathbf{r}) = 0 \quad (4)$$

(2,3)

and this can be satisfied for all  $\mathbf{r}$  only if

$${}^N\boldsymbol{\alpha}^{N'} = 0 \quad (5)$$

and

$${}^N\boldsymbol{\omega}^{N'} = 0 \quad (6)$$

in which event

$${}^N\mathbf{a}^P = {}^{N'}\mathbf{a}^P \quad (7)$$

(1,3,6)

and, therefore [see Eqs. (4.11.1) and (4.11.3)],

$$\tilde{F}_r^* = \tilde{F}'_r^* \quad (r = 1, \dots, p) \quad (8)$$

which means that whenever  $\tilde{F}_r + \tilde{F}_r^* = 0$  ( $r = 1, \dots, p$ ), then  $\tilde{F}'_r + \tilde{F}'_r^* = 0$  ( $r = 1, \dots, p$ ) and, consequently, that  $N'$  is a Newtonian reference frame.

To show that  $N'$  is not a Newtonian reference frame unless the acceleration in  $N$  of every point of  $N'$  is equal to zero, let  $O$  be a point of  $N'$  such that

$${}^N\mathbf{a}^O \neq 0 \quad (9)$$

and let  $S$  consist of a single particle  $P$  situated permanently at  $O$ . Then

$${}^{N'}\mathbf{a}^P = 0 \quad (10)$$

while

$${}^N\mathbf{a}^P = \underset{(1,2) (10)}{0} + \underset{(5,6) (9)}{{}^N\mathbf{a}^O} + 0 \neq 0 \quad (11)$$

so that

$$\underset{(10,11)}{{}^N\mathbf{a}^P} \neq \underset{(10,11)}{{}^{N'}\mathbf{a}^P} \quad (12)$$

and, therefore,

$$\tilde{F}_r^* \neq \tilde{F}_r^{*'} \quad (r = 1, \dots, p) \quad (13)$$

which means that whenever  $\tilde{F}_r + \tilde{F}_r^* = 0$  ( $r = 1, \dots, p$ ), then  $\tilde{F}_r' + \tilde{F}_r^{*'} \neq 0$  ( $r = 1, \dots, p$ ) and, consequently, that  $N'$  is not a Newtonian reference frame.

Returning to the question raised at the beginning of the section, we let  $A$  be a reference frame in which the Sun  $S$  remains fixed and relative to which the orientation of the reference frame  $F$  introduced in the example in Sec. 6.1 does not vary, so that  ${}^A\boldsymbol{\omega}^F = 0$ , but in which the center  $E^*$  of the Earth  $E$  moves on a plane curve.  $A$ ,  $S$ ,  $F$ , and  $E$  are depicted in Fig. 6.2.1, which also shows the Moon  $M$ .

Suppose that  $A$  is a Newtonian reference frame. Then  $F$  cannot possibly be a Newtonian reference frame, because there exists no point of  $F$  whose acceleration in  $A$  is equal to zero. In fact, the acceleration in  $A$  of every point that is fixed in  $F$  is equal to the acceleration  ${}^A\mathbf{a}^{E^*}$  of  $E^*$  in  $A$ , and this acceleration has a magnitude that can be estimated with sufficient accuracy for present purposes by assuming that  $E^*$  moves in  $A$  on a circle of radius  $R \approx 1.5 \times 10^{11}$  m, traced out once per year. How important is this acceleration? That depends on the acceleration in  $F$  of the particles of the system under consideration. As the angular velocity of  $F$  in  $A$  is equal to zero, and, hence, the angular acceleration of  $F$  in  $A$  is also equal to zero, the acceleration  ${}^A\mathbf{a}^P$  of a particle  $P$  in  $A$  differs from the acceleration  ${}^F\mathbf{a}^P$  of  $P$  in  $F$  by precisely  ${}^A\mathbf{a}^{E^*}$ . Errors resulting from regarding  $F$ , rather than  $A$ ,

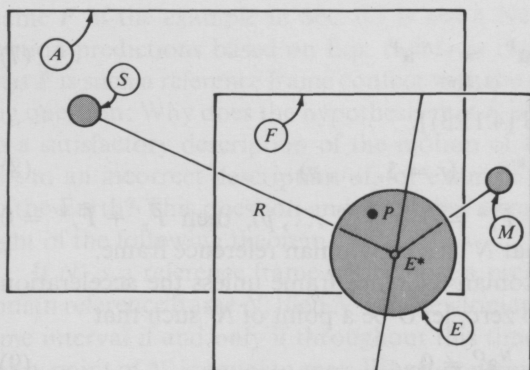


Figure 6.2.1



as a Newtonian reference frame therefore grow in importance as the ratio of  $|{}^A\mathbf{a}^{E*}|$  to  $|{}^F\mathbf{a}^P|$  increases. Hence, in connection with studies of motions of the Moon, an object whose particles move in  $F$ , essentially, on circles having radii of approximately  $4.0 \times 10^8$  m, each particle completing about 12 such orbits per year, it must make a substantial difference whether  $A$  or  $F$  is assigned the role of Newtonian reference frame, because the ratio in question here has the value (let  ${}^F\mathbf{a}^M$  be the acceleration in  $F$  of a typical particle of the Moon)

$$\frac{|{}^A\mathbf{a}^{E*}|}{|{}^F\mathbf{a}^M|} = \frac{1.5 \times 10^{11}}{4.0 \times 10^8} \left(\frac{1}{12}\right)^2 \approx 2.6 \quad (14)$$

which means that  $|{}^A\mathbf{a}^{E*}|$  cannot be regarded as negligible in comparison with  $|{}^F\mathbf{a}^M|$ . By way of contrast, for the pendulum considered in Sec. 6.1,  $g\alpha$  is a reasonable upper bound for  $|{}^F\mathbf{a}^P|$ , where  $\alpha$  is the maximum value of the angle between lines  $OQ$  and  $PQ$  in Fig. 6.1.1, so that, with  $g = 9.81 \text{ m/s}^2$  and  $\alpha$  even as small as  $0.01$  rad,

$$\frac{|{}^A\mathbf{a}^{E*}|}{|{}^F\mathbf{a}^P|} \approx 0.06 \quad (15)$$

Thus, so far as numerical results are concerned, it here matters far less whether  $A$  or  $F$  is regarded as a Newtonian reference frame.

As was pointed out in Sec. 6.1, the pendulum experiments performed by Foucault support the hypothesis that  $F$  is a Newtonian reference frame. Now [see Eq. (15)] we see that these experiments support the same hypothesis for  $A$ , but that comparisons of actual with predicted motions of the Moon can reveal which of these two reference frames is the stronger contender for the title of "true" Newtonian reference frame. It turns out that  $A$  is the winner of this contest. But this is not to say either that  $A$  actually is a Newtonian reference frame or that Eqs. (6.1.1) or (6.1.2) should be used only in conjunction with  $A$  (rather than with  $F$ ). Phenomena such as the nutation of the Earth and the motion of the Sun relative to the galaxy as a whole show that  $A$  falls short of perfection. Moreover, the use of  $A$  in place of  $F$  is desirable only when it makes a discernible difference; otherwise it merely complicates matters. Similarly, treating the Earth, rather than  $A$  or  $F$ , as a Newtonian reference frame is sound practice whenever doing so leads to analytical simplifications unaccompanied by significant losses in accuracy, as is the case in a large number of situations encountered in engineering. Hence, unless explicitly exempted, every use of Eqs. (6.1.1) or (6.1.2) in the remainder of this book will be predicated on the assumption that all reference frames rigidly attached to the Earth, as well as all reference frames all of whose points have zero acceleration relative to the Earth, may be regarded as Newtonian reference frames.

### 6.3 ADDITIONAL DYNAMICAL EQUATIONS

When generalized speeds are introduced in addition to  $u_1, \dots, u_p$  for the purpose of bringing into evidence forces and/or torques that contribute nothing to the generalized active forces  $\tilde{F}_1, \dots, \tilde{F}_p$  or, if  $S$  is a holonomic system, to  $F_1, \dots, F_n$ ,

then the dynamical equations corresponding to the additional generalized speeds furnish the needed information about the forces and/or torques in question. To generate these dynamical equations, follow the procedures set forth in Secs. 4.9 and 4.11 to form expressions for the generalized active forces and generalized inertia forces corresponding to the additional generalized speeds, and then substitute into Eqs. (6.1.1) or (6.1.2).

**Example** When the bearing surface  $B$  and the rod  $R$  of the system introduced in the example in Sec. 4.9 and depicted in Fig. 4.9.1 both are perfectly smooth, the dynamical equations governing the motion of the system, written by substituting from Eqs. (4.9.6), (4.9.7), (4.11.35), and (4.11.36) into Eqs. (6.1.1), are

$$\dot{u}_1 = g \cos \beta + q_1(u_2 \sin \beta)^2 \quad (1)$$

$$\dot{u}_2 = \frac{T \csc^2 \beta - 2mq_1 u_1 u_2}{mq_1^2 + 4ML^2/3} \quad (2)$$

and these, together with the kinematical equation

$$\dot{q}_1 = u_1 \quad (4.9.2) \quad (3)$$

permit one to determine  $q_1$ ,  $u_1$ , and  $u_2$  for  $t > 0$  when the values of these variables are known for  $t = 0$  (and  $T$  has been specified as a function of  $q_1$ ,  $u_1$ ,  $u_2$ , and  $t$ ). But if, as in the example in Sec. 4.10, the contact between the sleeve  $S$  and the bearing surface  $B$ , as well as the contact between  $P$  and  $R$ , is presumed to take place across a rough surface, then Eqs. (4.10.39) and (4.10.40) replace Eqs. (4.9.6) and (4.9.7), respectively, with the result that the dynamical equations become

$$\dot{u}_1 = g \cos \beta - \left( \frac{\mu_1'}{m} \right) (\rho_2^2 + \rho_3^2)^{1/2} \operatorname{sgn} u_1 + q_1 (u_2 \sin \beta)^2 \quad (4)$$

$$\dot{u}_2 = \frac{[T - (2\pi n^*/3)(b_2^3 - b_1^3)\mu_2' \operatorname{sgn} u_2] \csc^2 \beta - 2mq_1 u_1 u_2}{mq_1^2 + 4ML^2/3} \quad (5)$$

and, since these contain  $\rho_2$ ,  $\rho_3$ , and  $n^*$ , they do not suffice [together with Eq. (3)] for the determination of  $q_1$ ,  $u_1$ , and  $u_2$ . However, one can supplement Eqs. (3)–(5) with the dynamical equations corresponding to  $u_3$ ,  $u_5$ , and  $u_9$  as introduced in Sec. 4.9 [see Eqs. (4.9.11), (4.9.15), and (4.9.24), respectively]. These equations are

$$(m + M)g - \pi n^*(b_2^2 - b_1^2) - m\dot{u}_1 \cos \beta = 0 \quad (4.10.41) \quad (4.11.37) \quad (6.1.1) \quad (6)$$

$$\rho_3 - m(\dot{u}_2 q_1 + 2u_1 u_2) \sin \beta = 0 \quad (4.10.42) \quad (4.11.39) \quad (6.1.1) \quad (7)$$

$$-mg \sin \beta + \rho_2 + mu_2^2 q_1 \sin \beta \cos \beta = 0 \quad (4.10.43) \quad (4.11.40) \quad (6.1.1) \quad (8)$$

and with their aid one can eliminate  $n^*$ ,  $\rho_2$ , and  $\rho_3$  from Eqs. (4) and (5) and thus come into position to determine  $q_1$ ,  $u_1$ , and  $u_2$  as functions of  $t$ .

## 6.4 LINEARIZATION OF DYNAMICAL EQUATIONS

Frequently, one can obtain much useful information about the behavior of a system  $S$  from *linearized* forms of kinematical and/or dynamical equations, that is, equations derived from nonlinear equations by omitting all terms of second or higher degree in *perturbations* of some (or all) of the generalized speeds  $u_1, \dots, u_n$ , and generalized coordinates  $q_1, \dots, q_n$ . This is true primarily because linear differential equations generally can be solved more easily than can nonlinear differential equations. Of course, the solutions of such linearized equations lead one only to approximations of the solutions of the corresponding full, nonlinear equations; and they may be rather poor approximations. In any event, however, the approximations become ever better as the perturbations involved in the linearization take on ever smaller values.

When nonlinear kinematical and/or dynamical equations are in hand, one forms their linearized counterparts by expanding all functions of the perturbations involved in the linearization into power series in these perturbations and dropping all nonlinear terms. To formulate linearized dynamical equations directly, that is, without first writing exact dynamical equations, proceed as follows:

Develop fully nonlinear expressions for angular velocities of rigid bodies belonging to  $S$ , for velocities of mass centers of such bodies, and for velocities of particles of  $S$  to which contact and/or distance forces contributing to generalized active forces are applied. Use these nonlinear expressions to determine partial angular velocities and partial velocities by inspection. Linearize all angular velocities of rigid bodies and velocities of particles, and use the linearized forms to construct linearized angular accelerations and accelerations. Linearize all partial angular velocities and partial velocities. Form linearized generalized active forces and linearized generalized inertia forces, and substitute into Eqs. (6.1.1) or (6.1.2).

**Examples** As was shown in the example in Sec. 6.1, all motions of the Foucault pendulum are governed by the equations

$$\dot{u}_1 = 2\omega u_2(c_1c_2c\phi - s_2s\phi) + gc_1s_2 \quad (1)$$

(6.1.29)

$$\dot{u}_2 = -2\omega u_1(c_1c_2c\phi - s_2s\phi) - gs_1 \quad (2)$$

(6.1.30)

$$\dot{q}_1 = \frac{u_2}{Lc_2} \quad \dot{q}_2 = -\frac{u_1}{L} \quad (3)$$

(6.1.34)      (6.1.34)

As may be verified by inspection, these equations possess the solution

$$u_1 = u_2 = q_1 = q_2 = 0 \quad (4)$$

Hence, one may hope to obtain useful information regarding the motion of the pendulum by employing equations resulting from linearization of Eqs. (1)–(3) “about this solution,” that is, by replacing  $u_1$ ,  $u_2$ ,  $q_1$ , and  $q_2$  with perturbation functions  $u_1^*$ ,  $u_2^*$ ,  $q_1^*$ , and  $q_2^*$ , respectively, and then linearizing in these perturbations. For example, when  $u_1^*$  and  $u_2^*$  are written in place of  $u_1$  and  $u_2$ , respectively, in Eq. (1), and  $c_1$ ,  $c_2$ , and  $s_2$  are replaced with 1, 1, and  $q_2^*$ , respectively, these being the terms of degree less than 2 in the expansions of  $\cos q_1^*$ ,  $\cos q_2^*$ , and  $\sin q_2^*$ , respectively, then  $\dot{u}_1^*$  is given by

$$\dot{u}_1^* = 2\omega u_2^*(c\phi - q_2^*s\phi) + gq_2^* \quad (5)$$

(1)

This, however, is an incompletely linearized equation because  $u_2^*q_2^*$  is a second-degree term. Dropping this term, one arrives at the linear equation corresponding to Eq. (1), namely,

$$\dot{u}_1^* = 2\omega u_2^*c\phi + gq_2^* \quad (6)$$

Similarly, the linear equations corresponding to Eqs. (2) and (3) are found to be

$$\dot{u}_2^* = -2\omega u_1^*c\phi - gq_1^* \quad (7)$$

(2)

and

$$\dot{q}_1^* = \frac{u_2^*}{L} \quad \dot{q}_2^* = -\frac{u_1^*}{L} \quad (8)$$

(3)                      (3)

To solve Eqs. (6)–(8), and thus to obtain a detailed, approximate, analytical description of the motion of the pendulum, it is helpful to introduce  $\mathbf{p}$  as the linearized form of the vector  $\mathbf{r}$  shown in Fig. 6.1.4, that is, to let

$$\mathbf{p} = L(-q_2^*\mathbf{e}_1 + q_1^*\mathbf{e}_2) \quad (9)$$

(6.1.37)

for it may be verified by carrying out the indicated differentiations that Eqs. (6)–(8) are together equivalent to the single linear vector differential equation

$$\frac{{}^E d^2 \mathbf{p}}{dt^2} + 2\omega c\phi \mathbf{e}_3 \times \frac{{}^E d\mathbf{p}}{dt} + \frac{g}{L} \mathbf{p} = 0 \quad (10)$$

and this equation can be replaced with an even simpler one through the introduction of a reference frame  $R$  whose angular velocity in  $E$  is taken to be

$${}^E \boldsymbol{\omega}^R \triangleq -\omega c\phi \mathbf{e}_3 \quad (11)$$

which permits one to write

$$\frac{{}^E d\mathbf{p}}{dt} \stackrel{(2.3.1)}{=} \frac{{}^R d\mathbf{p}}{dt} + {}^E \boldsymbol{\omega}^R \times \mathbf{p} = \frac{{}^R d\mathbf{p}}{dt} - \omega c\phi \mathbf{e}_3 \times \mathbf{p} \quad (12)$$

(11)

$$\begin{aligned} \frac{{}^E d^2 \mathbf{p}}{dt^2} &\stackrel{(12)}{=} \frac{{}^R d^2 \mathbf{p}}{dt^2} - \omega c\phi \mathbf{e}_3 \times \frac{{}^R d\mathbf{p}}{dt} - \omega c\phi \mathbf{e}_3 \times \frac{{}^R d\mathbf{p}}{dt} \\ &\quad + (\omega c\phi)^2 \mathbf{e}_3 \times (\mathbf{e}_3 \times \mathbf{p}) \end{aligned} \quad (13)$$

(11)

Substitution from Eqs. (12) and (13) into Eq. (10) yields

$$\frac{{}^R d^2 \mathbf{p}}{dt^2} + \frac{g}{L} \mathbf{p} = 0 \quad (14)$$

if, as in Sec. 6.1, terms involving  $\omega^2$  are dropped. Now, Eq. (14) possesses the general solution

$$\mathbf{p} = \mathbf{A} \cos pt + \mathbf{B} \sin pt \quad (15)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are vectors fixed in reference frame  $R$ , and  $p$ , known as the *circular natural frequency* of the pendulum, is defined as

$$p = \left( \frac{g}{L} \right)^{1/2} \quad (16)$$

The vectors  $\mathbf{A}$  and  $\mathbf{B}$  depend on initial conditions. Specifically, denoting the initial values of  $\mathbf{p}$  and  ${}^R d\mathbf{p}/dt$  by  $\mathbf{p}(0)$  and  ${}^R d\mathbf{p}(0)/dt$ , respectively, one can write

$$\mathbf{A} = \underset{(15)}{\mathbf{p}(0)} \quad \mathbf{B} = \frac{1}{\underset{(15)}{p}} \frac{{}^R d\mathbf{p}(0)}{dt} \quad (17)$$

Suppose, for example, that  $P$  is initially displaced toward the east in such a way that the string makes an angle  $\alpha$  with the vertical, and that  $P$  is then released from a state of rest in  $E$ . Correspondingly,

$$q_1(0) = \alpha \quad q_2(0) = 0 \quad u_1(0) = u_2(0) = 0 \quad (18)$$

Hence,

$$\underset{(6.1.37,18)}{\mathbf{r}(0)} = L s \alpha \mathbf{e}_2(0) \quad (19)$$

so that linearization in  $\alpha$  yields

$$\mathbf{p}(0) = L \alpha \mathbf{e}_2(0) \quad (20)$$

where  $\mathbf{e}_2(0)$  denotes the value of  $\mathbf{e}_2$  in  $R$  at  $t = 0$ . Similarly, one can write

$$\begin{aligned} \frac{{}^R d\mathbf{r}(0)}{dt} &= \frac{{}^E d\mathbf{r}(0)}{dt} + {}^R \boldsymbol{\omega}^E(0) \times \mathbf{r}(0) \\ &= \underset{(11)}{{}^E \mathbf{v}^P(0)} + \omega c \phi \underset{(19)}{\mathbf{e}_3(0)} \times [L s \alpha \mathbf{e}_2(0)] \\ &= \underset{(6.1.33,6.1.34,18)}{0} - L \omega c \phi s \alpha \mathbf{e}_1(0) \end{aligned} \quad (21)$$

and, linearizing in  $\alpha$ , one has

$$\frac{{}^R d\mathbf{p}(0)}{dt} = \underset{(21)}{-L \omega c \phi s \alpha \mathbf{e}_1(0)} \quad (22)$$

Thus,  $\mathbf{p}$  can be written

$$\begin{aligned} \mathbf{p} &= \underset{(15)}{L\alpha} \underset{(17,20)}{\mathbf{e}_2(0)} \cos pt - L \underset{(17,22)}{\left(\frac{\omega}{p}\right)} c\phi \alpha \mathbf{e}_1(0) \sin pt \\ &= L\alpha \left[ -\left(\frac{\omega}{p}\right) c\phi \sin pt \mathbf{e}_1(0) + \cos pt \mathbf{e}_2(0) \right] \end{aligned} \quad (23)$$

According to Eq. (23),  $P^*$  (see Fig. 6.1.4) moves in  $R$  on an elliptical path having the proportions shown in Fig. 6.4.1, and  $P^*$  traverses this path once every  $2\pi/p$  units of time. The ellipse, being fixed in  $R$ , rotates in  $E$  at a rate of  $\omega c\phi$  radians per unit of time [see Eq. (11)], the rotation being clockwise as seen by an observer looking from point  $Q$  toward point  $O$  (see Fig. 6.1.4), provided that  $O$  is situated in the northern hemisphere. These predictions agree qualitatively with those obtained in Sec. 6.1 by using the full, nonlinear equations of motion, Eqs. (6.1.29), (6.1.30), and (6.1.34). To assess the merits of the linear theory in quantitative terms, one can compare  $|\mathbf{r}|$ —calculated by using Eq. (6.1.37) after integrating Eqs. (6.1.29), (6.1.30), and (6.1.34) numerically with  $\phi = 45^\circ$ ,  $L = 10$  m,  $\omega = 7.29 \times 10^{-5}$  rad/s,  $q_1(0) = \alpha$ ,  $q_2(0) = u_1(0) = u_2(0) = 0$ —with  $|\mathbf{p}|$  as obtained by using Eq. (23). Plots resulting from such calculations are shown in Fig. 6.4.2 for  $\alpha = 10^\circ$ ,  $30^\circ$ , and  $60^\circ$ . The solid curves represent values of  $|\mathbf{r}|$ , whereas the broken curves correspond to values of  $|\mathbf{p}|$ . As was to be expected, the agreement between the two curves associated with a given value of  $\alpha$  is best for  $\alpha = 10^\circ$  and worst for

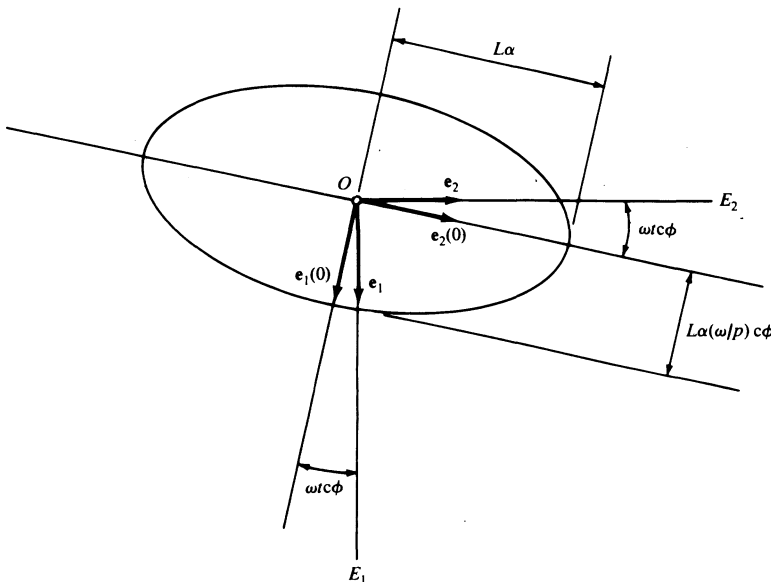


Figure 6.4.1

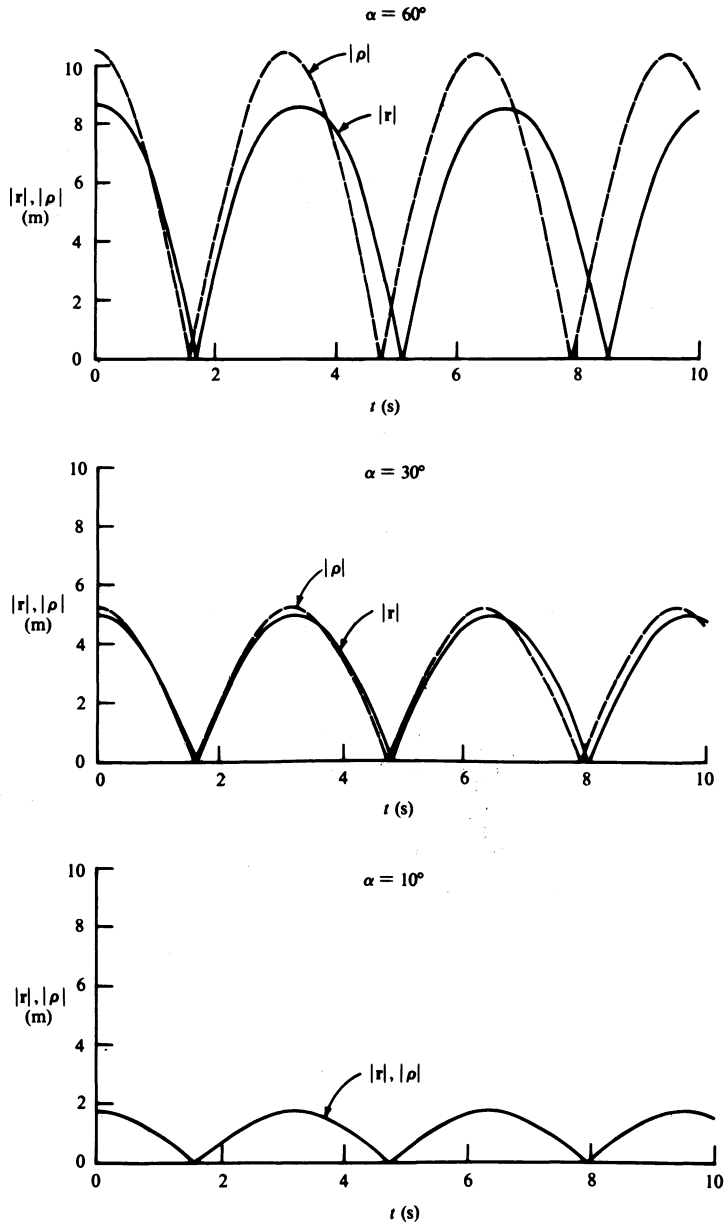


Figure 6.4.2

$\alpha = 60^\circ$ . Of course, whether or not any given result obtained from linearized equations can be regarded as “good” depends on the user’s needs.

To illustrate the process of formulating linearized equations of motion without first constructing fully nonlinear equations, we consider a uniform bar  $B$  of mass  $m$  supported by frictionless sliders on a circular wire  $W$  of radius  $R$ , as shown in Fig. 6.4.3, and suppose that  $W$  is being made to rotate with a constant angular speed  $\Omega$  about a fixed vertical axis passing through the center  $O$  of  $W$ . Under these circumstances, it seems reasonable to suppose that  $B$  can remain at rest relative to  $W$ , that is, that  $q_1$  (see Fig. 6.4.3) can have a constant value, say,  $\bar{q}_1$ , so that the generalized speed  $u_1$  defined as

$$u_1 \triangleq \dot{q}_1 \quad (24)$$

remains equal to zero; and one can undertake the formulation of linear equations governing perturbations  $q_1^*$  and  $u_1^*$  introduced by writing

$$q_1 = \bar{q}_1 + q_1^* \quad u_1 = 0 + u_1^* \quad (25)$$

One such equation is available immediately, namely,

$$\dot{q}_1^* = u_1^* \quad (25) \quad (24) \quad (25) \quad (26)$$

Whether or not  $B$  can, in fact, move as postulated, that is, whether or not there exist real values of  $\bar{q}_1$ , will be discussed once the linearized dynamical equation governing  $u_1^*$  and  $q_1^*$  has been formulated.

The angular velocity  $\omega$  of  $B$  and the velocity  $\mathbf{v}^*$  of the center  $B^*$  of  $B$  are given by (see Fig. 6.4.3 for the unit vectors  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , and  $\mathbf{b}_3$ )

$$\omega = \Omega(s_1 \mathbf{b}_1 + c_1 \mathbf{b}_2) + u_1 \mathbf{b}_3 \quad (27)$$

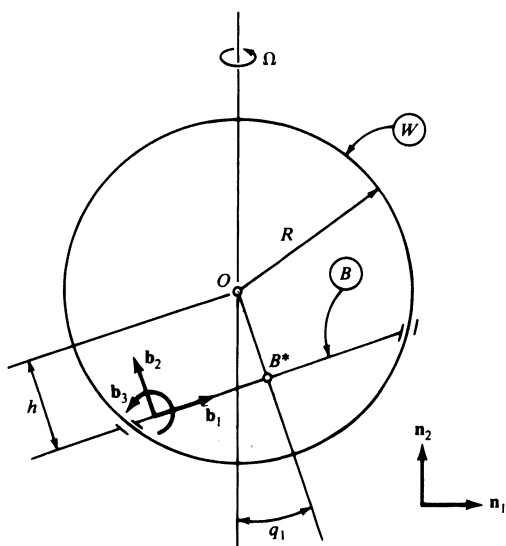


Figure 6.4.3



and

$$\mathbf{v}^* = h(u_1 \mathbf{b}_1 - \Omega s_1 \mathbf{b}_3) \quad (28)$$

respectively. Hence, the partial angular velocity  $\omega_1$  and partial velocity  $\mathbf{v}_1^*$  are

$$\omega_1 = \mathbf{b}_3 \quad \mathbf{v}_1^* = h \mathbf{b}_1 \quad (29)$$

$$(27) \quad (28)$$

To linearize Eqs. (27) and (28) in  $q_1^*$  and  $u_1^*$ , we observe that

$$s_1 = \sin(\bar{q}_1 + q_1^*) = \sin \bar{q}_1 \cos q_1^* + \cos \bar{q}_1 \sin q_1^* \quad (30)$$

$$(25)$$

so that, if  $\bar{s}_1$  and  $\bar{c}_1$  are written in place of  $\sin \bar{q}_1$  and  $\cos \bar{q}_1$ , respectively, and  $\cos q_1^*$  and  $\sin q_1^*$  are replaced with unity and  $q_1^*$ , respectively, then

$$s_1 \approx \bar{s}_1 + \bar{c}_1 q_1^* \quad (31)$$

$$(30)$$

Similarly,

$$c_1 \approx \bar{c}_1 - \bar{s}_1 q_1^* \quad (32)$$

Hence, the linearized forms of  $\omega$  and  $\mathbf{v}^*$  are

$$\omega = \Omega[(\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{b}_1 + (\bar{c}_1 - \bar{s}_1 q_1^*) \mathbf{b}_2] + u_1^* \mathbf{b}_3 \quad (33)$$

$$(27)$$

$$(31)$$

$$(32)$$

$$(25)$$

and

$$\mathbf{v}^* \approx h[u_1^* \mathbf{b}_1 - \Omega(\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{b}_3] \quad (34)$$

$$(28)$$

$$(25)$$

$$(31)$$

As for  $\omega_1$  and  $\mathbf{v}_1^*$ , these, as written in Eqs. (29), contain no terms that are nonlinear in  $q_1^*$ . Note, however, that  $\mathbf{v}_1^*$  can be written

$$\mathbf{v}_1^* = h(c_1 \mathbf{n}_1 + s_1 \mathbf{n}_2) \quad (35)$$

$$(29)$$

where  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are unit vectors directed as shown in Fig. 6.4.3, and that the linearized form of  $\mathbf{v}_1^*$  then is

$$\mathbf{v}_1^* \approx h[(\bar{c}_1 - \bar{s}_1 q_1^*) \mathbf{n}_1 + (\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{n}_2] \quad (36)$$

$$(35)$$

$$(32)$$

$$(31)$$

To show that, in general, one must form partial velocities (and partial angular velocities) *before* linearizing, we rewrite Eq. (34) as

$$\mathbf{v}^* \approx h[u_1^*(\bar{c}_1 - \bar{s}_1 q_1^*) \mathbf{n}_1 + u_1^*(\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{n}_2 - \Omega(\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{b}_3] \quad (37)$$

or, after completing the linearization process,

$$\mathbf{v}^* \approx h[u_1^* \bar{c}_1 \mathbf{n}_1 + u_1^* \bar{s}_1 \mathbf{n}_2 - \Omega(\bar{s}_1 + \bar{c}_1 q_1^*) \mathbf{b}_3] \quad (38)$$

$$(37)$$

Starting with this equation, one would conclude that  $\mathbf{v}_1^* \approx h(\bar{c}_1 \mathbf{n}_1 + \bar{s}_1 \mathbf{n}_2)$ , which conflicts with Eq. (36) and is incorrect.

Linearized expressions for the angular acceleration  $\alpha$  of  $B$  and the acceleration  $\mathbf{a}^*$  of  $B^*$  are formed by working with the already linearized Eqs. (33) and (34), respectively. Specifically,

$$\alpha \approx \underset{(33)}{\Omega u_1^*} (\underset{(26)}{\bar{c}_1 \mathbf{b}_1} - \bar{s}_1 \mathbf{b}_2) + \dot{u}_1^* \mathbf{b}_3 \quad (39)$$

and

$$\mathbf{a}^* \approx \underset{(34)}{h(\dot{u}_1^* \mathbf{b}_1} - \underset{(26)}{\Omega \bar{c}_1 u_1^*} \mathbf{b}_3) + \omega \times \mathbf{v}^* \quad (40)$$

Now,

$$\begin{aligned} \omega \times \mathbf{v}^* &\underset{(33,34)}{\approx} -h\Omega^2(\bar{c}_1 - \bar{s}_1 q_1^*)(\bar{s}_1 + \bar{c}_1 q_1^*)\mathbf{b}_1 + \cdots \\ &\approx -h\Omega^2[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*]\mathbf{b}_1 + \cdots \end{aligned} \quad (41)$$

where only the  $\mathbf{b}_1$  component has been worked out in detail because  $\mathbf{v}_1^*$  [see Eqs. (29)] is parallel to  $\mathbf{b}_1$ , and  $\mathbf{v}_1^*$  presently will be dot-multiplied with  $\mathbf{a}^*$  [see Eqs. (4.11.7) and (4.11.6)]. Hence, we have

$$\mathbf{a}^* \underset{(40,41)}{\approx} h\{\dot{u}_1^* - \Omega^2[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*]\}\mathbf{b}_1 + \cdots \quad (42)$$

The inertia torque  $\mathbf{T}^*$  for  $B$  is given by

$$\begin{aligned} \mathbf{T}^* &\underset{(4.11.10)}{\approx} -\frac{m(R^2 - h^2)}{3} [\underset{(39)}{\dot{u}_1^*} + \Omega^2(\bar{s}_1 + \underset{(33)}{\bar{c}_1 q_1^*})(\bar{c}_1 - \bar{s}_1 q_1^*)]\mathbf{b}_3 + \cdots \\ &\approx -\frac{m(R^2 - h^2)}{3} \{\dot{u}_1^* + \Omega^2[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*]\}\mathbf{b}_3 + \cdots \end{aligned} \quad (43)$$

where, once again, only the term that will survive [after dot-multiplication, as per Eq. (4.11.7), with  $\omega_1$  as given in Eqs. (29)] has been written in detail. Now we are in position to form the generalized inertia force  $F_1^*$  as

$$\begin{aligned} F_1^* &\underset{(4.11.7)}{=} \omega_1 \cdot \mathbf{T}^* + \mathbf{v}_1^* \cdot \underset{(4.11.6)}{(-m\mathbf{a}^*)} \\ &= -\frac{m(R^2 - h^2)}{3} \{\dot{u}_1^* + \underset{(29,43)}{\Omega^2[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*]}\} \\ &\quad - mh^2 \{\dot{u}_1^* - \underset{(29,42)}{\Omega^2[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*]}\} \\ &= -m \left\{ \left( \frac{R^2 - h^2}{3} + h^2 \right) \dot{u}_1^* \right. \\ &\quad \left. + \Omega^2 \left( \frac{R^2 - h^2}{3} - h^2 \right) [\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*] \right\} \end{aligned} \quad (44)$$

and the generalized active force  $F_1$  is given by

$$\begin{aligned} F_1 &= -mg(s_1 \mathbf{b}_1 + c_1 \mathbf{b}_2) \cdot \mathbf{v}_1^* = -mghs_1 \quad (29) \\ &\approx -mgh(\bar{s}_1 + \bar{c}_1 q_1^*) \quad (31) \end{aligned} \quad (45)$$

Finally, substitution from Eqs. (44) and (45) into Eqs. (6.1.2) yields

$$(R^2 + 2h^2)\dot{u}_1^* + \Omega^2(R^2 - 4h^2)[\bar{c}_1 \bar{s}_1 + (\bar{c}_1^2 - \bar{s}_1^2)q_1^*] + 3gh(\bar{s}_1 + \bar{c}_1 q_1^*) \approx 0 \quad (46)$$

Now, this equation must be satisfied by  $q_1^* = u_1^* = 0$ . Otherwise,  $q_1 = \bar{q}_1$  and  $u_1 = 0$  cannot be solutions of the full, nonlinear equations governing  $q_1$  and  $u_1$ . Hence, by setting  $q_1^* = u_1^* = 0$  in Eq. (46), we arrive at the conclusion that  $\bar{q}_1$  must satisfy the equation

$$\Omega^2(R^2 - 4h^2)\bar{c}_1 \bar{s}_1 + 3gh\bar{s}_1 = 0 \quad (47)$$

Moreover, when this equation is satisfied, Eq. (46) reduces to

$$\dot{u}_1^* + \frac{\Omega^2(R^2 - 4h^2)(\bar{c}_1^2 - \bar{s}_1^2) + 3gh\bar{c}_1}{R^2 + 2h^2} q_1^* \approx 0 \quad (48)$$

This is the desired linearized dynamical equation.

Equation (47) possesses the physically distinct solutions  $\bar{q}_1 = 0$  and  $\bar{q}_1 = \pi$  regardless of the values of  $\Omega$ ,  $R$ , and  $h$ ; when  $\bar{q}_1 \neq 0$  and  $\bar{q}_1 \neq \pi$ , Eq. (47) requires that the cosine of  $\bar{q}_1$  be given by

$$\bar{c}_1 = \frac{3gh}{\Omega^2(4h^2 - R^2)} \quad (49)$$

Now, real values of  $\bar{q}_1$  satisfying this equation exist if and only if

$$-1 \leq \frac{3gh}{\Omega^2(4h^2 - R^2)} \leq 1 \quad (50)$$

Thus,  $B$  can move as postulated, either with  $\bar{q}_1 = 0$  or  $\bar{q}_1 = \pi$  and  $\Omega$ ,  $R$ , and  $h$  unrestricted, or with  $\bar{q}_1$  governed by Eq. (49) and  $\Omega$ ,  $R$ , and  $h$  restricted by Eq. (50).

## 6.5 SYSTEMS AT REST IN A NEWTONIAN REFERENCE FRAME

Since rest is a special form of motion, Eqs. (6.1.1) and (6.1.2) apply to any system  $S$  at rest in a Newtonian reference frame  $N$ , reducing under these circumstances to

$$\tilde{F}_r = 0 \quad (r = 1, \dots, p) \quad (1)$$

and

$$F_r = 0 \quad (r = 1, \dots, n) \quad (2)$$

respectively. Moreover, if  $S$  possesses a potential energy  $V$  in  $N$  (see Sec. 5.1), then Eqs. (1) may be replaced with [see Eqs. (5.1.18)]

$$\sum_{s=1}^n \frac{\partial V}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) = 0 \quad (r = 1, \dots, p) \quad (3)$$

or with [see Eqs. (5.1.14)]

$$\frac{\partial V}{\partial q_r} + \sum_{s=p+1}^n \frac{\partial V}{\partial q_s} C_{sr} = 0 \quad (r = 1, \dots, p) \quad (4)$$

whereas Eqs. (2) are equivalent to [see Eqs. (5.1.9)]

$$\sum_{s=1}^n \frac{\partial V}{\partial q_s} W_{sr} = 0 \quad (r = 1, \dots, n) \quad (5)$$

or to [see Eqs. (5.1.2)]

$$\frac{\partial V}{\partial q_r} = 0 \quad (r = 1, \dots, n) \quad (6)$$

The last set of equations expresses the *principle of stationary potential energy*; Eqs. (3)–(5) represent generalizations of this principle.

**Example** Figure 6.5.1 shows a frame  $F$  supported by wheels  $W_1$ ,  $W_2$ , and  $W_3$ , this assembly resting on a plane  $P$  that is inclined at an angle  $\theta$  to the horizontal.  $W_1$  can rotate freely relative to  $F$  about the axis of  $W_1$ ; rotation of  $W_2$  relative to  $F$  is resisted by a braking couple whose torque has a magnitude  $T$ ; and  $W_3$  is mounted in a fork that can rotate freely relative to  $F$  about the axis of the fork, which is normal to  $P$ , while  $W_3$  is free to rotate relative to the fork about the axis of  $W_3$ .

The configuration of the system  $S$  formed by the frame, the fork, and the wheels is characterized completely by wheel rotation angles  $q_1$ ,  $q_2$ , and  $q_3$ ,

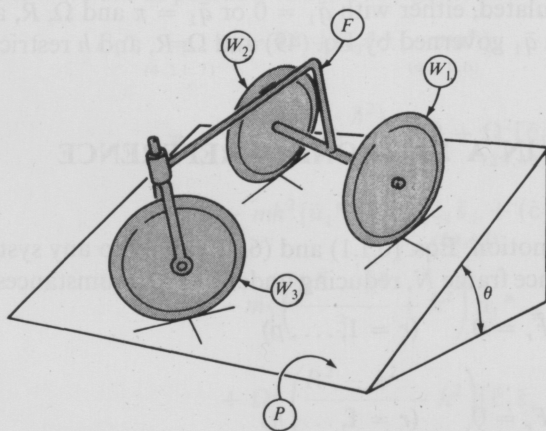


Figure 6.5.1

a steering angle  $q_4$ , a frame orientation angle  $q_5$ , and two position coordinates,  $q_6$  and  $q_7$ , defined as

$$q_6 \triangleq \mathbf{p}^* \cdot \mathbf{n}_1 \quad q_7 \triangleq \mathbf{p}^* \cdot \mathbf{n}_2 \quad (7)$$

where  $\mathbf{p}^*$  is the position vector from a point  $O$  fixed in  $P$  to the mass center  $S^*$  of  $S$ , and  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are, respectively, a horizontal unit vector and a unit vector pointing in the direction of steepest descent on  $P$ , as indicated in Fig. 6.5.2. Generalized speeds  $u_1, \dots, u_7$  may be defined as

$$u_1 \triangleq \dot{q}_3 \quad u_2 \triangleq \dot{q}_4 \quad u_3 \triangleq \dot{q}_5 \quad (8)$$

$$u_4 \triangleq \dot{q}_1 \quad u_5 \triangleq \dot{q}_2 \quad u_6 \triangleq \mathbf{v}^* \cdot \mathbf{f}_1 \quad u_7 \triangleq \mathbf{v}^* \cdot \mathbf{f}_2 \quad (9)$$

where  $\mathbf{v}^*$  is the velocity of  $S^*$ , and  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are unit vectors directed as indicated in Fig. 6.5.2.

On the basis of the assumption that  $W_1$ ,  $W_2$ , and  $W_3$  roll, rather than slip, on  $P$ , equations are to be formulated for the purpose of determining how  $T$  is related to the inclination angle  $\theta$ , the total mass  $M$  of  $S$ , the dimensions  $R$ ,  $a$ ,  $b$ , and  $L$  (see Fig. 6.5.2), and the generalized coordinates  $q_1, \dots, q_7$  when  $S$  is at rest.

The assumption that  $W_1$ ,  $W_2$ , and  $W_3$  do not slip on  $P$  leads to five motion constraint equations (see Sec. 2.13). Hence,  $S$  possesses two degrees of freedom, and Eqs. (1) can be written with  $p = 2$ . To form  $\tilde{F}_r$  ( $r = 1, 2$ ), we write

$$\tilde{F}_r = \underset{(4.6.1)}{Mg} \underset{(4.8.2)}{\mathbf{k}} \cdot \tilde{\mathbf{v}}_r^* + T \mathbf{f}_2 \cdot \tilde{\boldsymbol{\omega}}_r^F - T \mathbf{f}_2 \cdot \tilde{\boldsymbol{\omega}}_r^{W_2} \quad (r = 1, 2) \quad (10)$$

where  $\mathbf{k}$  is a unit vector directed vertically downward; that is, with  $\mathbf{f}_3 \triangleq \mathbf{f}_1 \times \mathbf{f}_2$ ,

$$\mathbf{k} = \sin q_5 \sin \theta \mathbf{f}_1 + \cos q_5 \sin \theta \mathbf{f}_2 - \cos \theta \mathbf{f}_3 \quad (11)$$

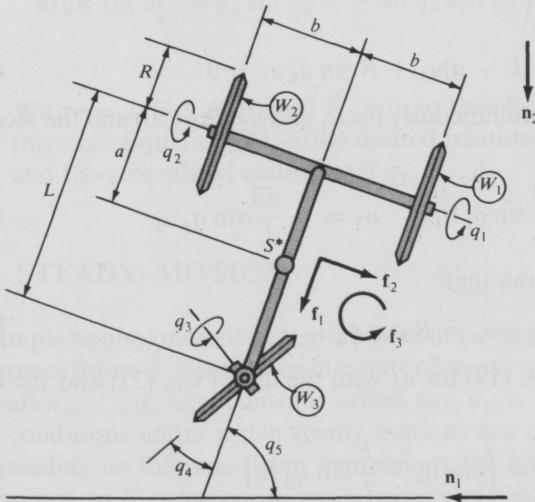


Figure 6.5.2

The partial velocities and partial angular velocities appearing in Eqs. (10) are found as follows.

The velocities of the points of  $W_1$ ,  $W_2$ , and  $W_3$  that are in contact with  $P$  must be equal to zero. In the case of  $W_1$ , this requirement can be expressed as [use Eq. (2.7.1) twice]

$$\mathbf{v}^* + \boldsymbol{\omega}^F \times (-a\mathbf{f}_1 + b\mathbf{f}_2) + \boldsymbol{\omega}^{W_1} \times (-R\mathbf{f}_3) = 0 \quad (12)$$

where

$$\boldsymbol{\omega}^F = u_3 \mathbf{f}_3 \quad (8)$$

and

$$\boldsymbol{\omega}^{W_1} = \boldsymbol{\omega}^F + \dot{q}_1 \mathbf{f}_2 = u_3 \mathbf{f}_3 + u_4 \mathbf{f}_2 \quad (14)$$

(2.4.1)                      (13)                      (9)

Moreover,

$$\mathbf{v}^* = u_6 \mathbf{f}_1 + u_7 \mathbf{f}_2 \quad (15)$$

(9)

Hence,

$$(u_6 - bu_3 - Ru_4)\mathbf{f}_1 + (u_7 - au_3)\mathbf{f}_2 = 0 \quad (16)$$

(15)    (13)    (14)                      (15)    (13)    (12)

which means that

$$u_6 - bu_3 - Ru_4 = 0 \quad u_7 - au_3 = 0 \quad (17)$$

Proceeding similarly in connection with  $W_2$  and  $W_3$ , one finds that

$$\boldsymbol{\omega}^{W_2} = u_3 \mathbf{f}_3 + u_5 \mathbf{f}_2 \quad \boldsymbol{\omega}^{W_3} = (u_3 - u_2)\mathbf{f}_3 + u_1(\sin q_4 \mathbf{f}_1 + \cos q_4 \mathbf{f}_2) \quad (18)$$

and that, in addition to Eqs. (17), the motion constraint equations

$$u_6 + bu_3 - Ru_5 = 0 \quad u_6 - R \cos q_4 u_1 = 0 \quad (19)$$

and

$$u_7 + (L - a)u_3 + R \sin q_4 u_1 = 0 \quad (20)$$

must be satisfied. Solved simultaneously for  $u_3$  and  $u_7$ , Eq. (20) and the second of Eqs. (17) yield

$$u_3 = -\frac{R}{L} \sin q_4 u_1 \quad u_7 = -\frac{aR}{L} \sin q_4 u_1 \quad (21)$$

The second of Eqs. (19) shows that

$$u_6 = R \cos q_4 u_1 \quad (22)$$

and, solving the first of Eqs. (19) for  $u_5$  with the aid of Eq. (22) and the first of Eqs. (21), one obtains

$$u_5 = \left( \cos q_4 - \frac{b}{L} \sin q_4 \right) u_1 \quad (23)$$

Hence,

$$\mathbf{v}^* = \underset{(15)}{R \cos q_4} \underset{(22)}{u_1 \mathbf{f}_1} - \underset{(21)}{\frac{aR}{L} \sin q_4} u_1 \mathbf{f}_2 \quad (24)$$

$$\underset{(13)}{\omega^F} = - \underset{(21)}{\frac{R}{L} \sin q_4} u_1 \mathbf{f}_3 \quad (25)$$

and

$$\underset{(21)}{\omega^{W_2}} = - \underset{(21)}{\frac{R}{L} \sin q_4} u_1 \mathbf{f}_3 + \left( \underset{(23)}{\cos q_4 - \frac{b}{L} \sin q_4} \right) u_1 \mathbf{f}_2 \quad (26)$$

The partial velocities of  $S^*$  and the partial angular velocities of  $F$  and  $W_2$  needed for substitution into Eqs. (10) thus are given by [see Eqs. (24)–(26)]

$$\tilde{\mathbf{v}}_1^* = R \cos q_4 \mathbf{f}_1 - \frac{aR}{L} \sin q_4 \mathbf{f}_2 \quad (27)$$

$$\tilde{\omega}_1^F = - \frac{R}{L} \sin q_4 \mathbf{f}_3 \quad (28)$$

$$\tilde{\omega}_1^{W_2} = \left( \cos q_4 - \frac{b}{L} \sin q_4 \right) \mathbf{f}_2 - \frac{R}{L} \sin q_4 \mathbf{f}_3 \quad (29)$$

and

$$\tilde{\mathbf{v}}_2^* = \tilde{\omega}_2^F = \tilde{\omega}_2^{W_2} = 0 \quad (30)$$

Substituting these into Eqs. (10) with  $r = 1$ , and setting the result equal to zero in accordance with Eqs. (1), one finds with the aid of Eq. (11) that

$$MgR \sin \theta \left( \cos q_4 \sin q_5 - \frac{a}{L} \sin q_4 \cos q_5 \right) - T \left( \cos q_4 - \frac{b}{L} \sin q_4 \right) = 0 \quad (31)$$

With  $r = 2$ , Eqs. (10) yield  $\tilde{F}_2 = 0$ , so that Eqs. (1) are satisfied identically in this case. Equation (31) is the desired relationship between  $T$ ,  $\theta$ ,  $M$ ,  $R$ ,  $a$ ,  $b$ ,  $L$ , and the generalized coordinates  $q_1, \dots, q_7$ .

## 6.6 STEADY MOTION

A simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a Newtonian reference frame  $N$  is said to be in a state of *steady motion* in  $N$  when the generalized speeds  $u_1, \dots, u_p$  have constant values, say,  $\bar{u}_1, \dots, \bar{u}_p$ , respectively. To determine the conditions under which steady motions can occur, use Eqs. (6.1.1) or (6.1.2), proceeding as follows: Form expressions for angular velocities of rigid bodies belonging to  $S$ , velocities of mass centers of these bodies, and so forth, *without*

regard to the fact that  $u_1, \dots, u_p$  are to remain constant, and use these expressions to construct partial angular velocities and partial velocities. Set  $u_r = \bar{u}_r$  ( $r = 1, \dots, p$ ) in angular velocity and velocity expressions, then differentiate with respect to time to generate needed angular accelerations of rigid bodies and accelerations of various points. Formulate expressions for  $\tilde{F}_r$  and  $\tilde{F}_r^*$  ( $r = 1, \dots, p$ ) in the case of Eqs. (6.1.1), or  $F_r$  and  $F_r^*$  ( $r = 1, \dots, n$ ) in the case of Eqs. (6.1.2), and substitute into Eqs. (6.1.1) or (6.1.2).

**Example** Figure 6.6.1 shows a right-circular, uniform, solid cone  $C$  in contact with a fixed, horizontal plane  $P$ . The motion that  $C$  performs—when  $C$  rolls on  $P$  in such a way that the mass center  $C^*$  of  $C$  (see Fig. 6.6.1) remains fixed while the plane determined by the axis of  $C$  and a vertical line passing through  $C^*$  has an angular velocity  $-\Omega\mathbf{k}$  ( $\Omega$  constant)—is a steady motion, as will be shown presently. But this motion can take place only if  $\Omega$ , the radius  $R$  of the base of  $C$ , the height  $4h$  of  $C$ , and the inclination angle  $\theta$  (see Fig. 6.6.1) are related to each other suitably. To determine the conditions under which the motion is possible, we begin by noting that  $C$  has three degrees of freedom, and introduce generalized speeds  $u_1, u_2$ , and  $u_3$  as

$$u_r \triangleq \boldsymbol{\omega} \cdot \mathbf{b}_r \quad (r = 1, 2, 3) \quad (1)$$

where  $\boldsymbol{\omega}$  is the angular velocity of  $C$ , and  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_3$  are mutually perpendicular unit vectors directed as indicated in Fig. 6.6.1. (Note that  $\mathbf{b}_2$  and  $\mathbf{b}_3$  are *not* fixed in  $C$ .)

As is pointed out in Problem 3.12,  $\boldsymbol{\omega}$  is given by

$$\boldsymbol{\omega} = \Omega s \theta \left( \frac{h}{R} \mathbf{b}_1 + \mathbf{b}_2 \right) \quad (2)$$

when  $C$  moves as required. Hence,  $u_1, u_2$ , and  $u_3$  then have the constant values

$$\begin{aligned} \bar{u}_1 &= \Omega s \theta \frac{h}{R} & \bar{u}_2 &= \Omega s \theta & \bar{u}_3 &= 0 \\ &_{(1)} &_{(1)} &_{(2)} &_{(1)(2)} \end{aligned} \quad (3)$$

respectively.

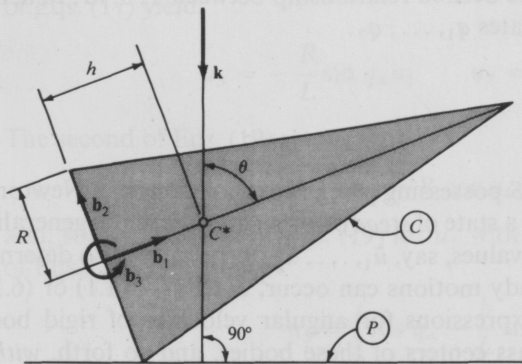


Figure 6.6.1



The generalized active forces  $\tilde{F}_r$  ( $r = 1, 2, 3$ ) are expressed most conveniently as

$$\tilde{F}_r = Mg\mathbf{k} \cdot \tilde{\mathbf{v}}_r^* \quad (r = 1, 2, 3) \quad (4)$$

(4.8.2)

where  $M$  is the mass of  $C$  and

$$\mathbf{k} = -(c\theta\mathbf{b}_1 + s\theta\mathbf{b}_2) \quad (5)$$

As for the generalized inertia forces  $\tilde{F}_r^*$  ( $r = 1, 2, 3$ ), we write

$$\tilde{F}_r^* = \tilde{\boldsymbol{\omega}}_r \cdot \mathbf{T}^* + \tilde{\mathbf{v}}_r^* \cdot \mathbf{R}^* \quad (r = 1, 2, 3) \quad (6)$$

(4.11.7)

and defer detailed consideration of  $\mathbf{T}^*$  and  $\mathbf{R}^*$  until after expressions for  $\tilde{\boldsymbol{\omega}}_r$  and  $\tilde{\mathbf{v}}_r^*$  have been constructed.

From Eqs. (1),

$$\boldsymbol{\omega} = u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3 \quad (7)$$

and, when  $C$  rolls on  $P$ , the velocity  $\mathbf{v}^*$  of  $C^*$  is given by

$$\begin{aligned} \mathbf{v}^* &= \boldsymbol{\omega} \times (h\mathbf{b}_1 + R\mathbf{b}_2) \\ &= -Ru_3\mathbf{b}_1 + hu_3\mathbf{b}_2 + (Ru_1 - hu_2)\mathbf{b}_3 \end{aligned} \quad (8)$$

(7)

The partial angular velocities and partial velocities obtained by inspection of Eqs. (7) and (8), respectively, are recorded in Table 6.6.1.

The steady motion form of  $\boldsymbol{\omega}$  is available in Eq. (2). Differentiating both sides of this equation with respect to time in order to find  $\boldsymbol{\alpha}$ , the angular acceleration of  $C$ , we obtain

$$\boldsymbol{\alpha} = \boldsymbol{\omega}^B \times \left[ \Omega s\theta \left( \frac{h}{R} \mathbf{b}_1 + \mathbf{b}_2 \right) \right] \quad (9)$$

where  $\boldsymbol{\omega}^B$ , the angular velocity of a reference frame in which  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , and  $\mathbf{b}_3$  are fixed, is given by

$$\boldsymbol{\omega}^B = \Omega(c\theta\mathbf{b}_1 + s\theta\mathbf{b}_2) \quad (10)$$

Consequently,

$$\boldsymbol{\alpha} = \Omega^2 s\theta \left( c\theta - \frac{h}{R} s\theta \right) \mathbf{b}_3 \quad (11)$$

(9,10)

**Table 6.6.1**

$r$	$\tilde{\boldsymbol{\omega}}_r$	$\tilde{\mathbf{v}}_r^*$
1	$\mathbf{b}_1$	$R\mathbf{b}_3$
2	$\mathbf{b}_2$	$-h\mathbf{b}_3$
3	$\mathbf{b}_3$	$-R\mathbf{b}_1 + h\mathbf{b}_2$

The velocity  $\mathbf{v}^*$  is equal to zero, by hypothesis, during the steady motion of interest, a result one can verify by setting  $u_r = \bar{u}_r$  ( $r = 1, 2, 3$ ) in Eq. (8) and then using Eqs. (3). It follows that  $\mathbf{a}^*$ , the acceleration of  $C^*$ , vanishes, and this means that the inertia force  $\mathbf{R}^*$  appearing in Eqs. (6) also vanishes. As for the inertia torque  $\mathbf{T}^*$ , one can express this as

$$\mathbf{T}^* = \Omega^2 s \theta \left( I_1 \frac{h}{R} s \theta - I_2 c \theta \right) \mathbf{b}_3 \quad (12)$$

by substituting from Eqs. (11) and (2) into Eq. (4.11.8), with

$$\mathbf{I} = I_1 \mathbf{b}_1 \mathbf{b}_1 + I_2 (\mathbf{b}_2 \mathbf{b}_2 + \mathbf{b}_3 \mathbf{b}_3) \quad (13)$$

Expressed in terms of  $M$ ,  $R$ , and  $h$ , the moments of inertia  $I_1$  and  $I_2$  are given by (see Appendix I)

$$I_1 = \frac{3MR^2}{10} \quad I_2 = \frac{3M(R^2 + 4h^2)}{20} \quad (14)$$

Referring to Eq. (5) and Table 6.6.1, one finds by substitution into Eqs. (4) that

$$\tilde{F}_1 = \tilde{F}_2 = 0 \quad \tilde{F}_3 = Mg(Rc\theta - hs\theta) \quad (15)$$

Similarly, Eqs. (6) lead to

$$\tilde{F}_1^* = \tilde{F}_2^* = 0 \quad \tilde{F}_3^* = \Omega^2 s \theta \left( I_1 \frac{h}{R} s \theta - I_2 c \theta \right) \quad (16)$$

Consequently, Eqs. (1) are satisfied identically when  $r = 1$  and  $r = 2$ ; for  $r = 3$ , substitution from Eqs. (15), (16), and (14) into Eqs. (1) leads to the conclusion that the steady motion under consideration can take place only when

$$\frac{3}{20} \left( \frac{R\Omega^2}{g} \right) \left\{ \left[ 1 + 4 \left( \frac{h}{R} \right)^2 \right] c\theta - 2 \left( \frac{h}{R} \right) s\theta \right\} s\theta + \frac{h}{R} s\theta - c\theta = 0$$

## 6.7 MOTIONS RESEMBLING STATES OF REST

A simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a Newtonian reference frame  $N$  is said to be performing a *motion resembling a state of rest* when the generalized coordinates  $q_1, \dots, q_n$  have constant values, say,  $\bar{q}_1, \dots, \bar{q}_n$ , respectively. To determine the conditions under which such motions are possible, one can use Eqs. (6.1.1) or (6.1.2), employing a procedure analogous to that set forth in Sec. 6.6. Alternatively, if the kinetic energy  $K$ , when regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , does not involve  $t$  explicitly, then one can use the equations

$$\sum_{s=1}^n \frac{\partial K}{\partial \dot{q}_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) + \tilde{F}_r = 0 \quad (r = 1, \dots, p) \quad (1)$$

where  $W_{sr}$  ( $s = 1, \dots, n; r = 1, \dots, p$ ) and  $A_{kr}$  ( $k = p + 1, \dots, n; r = 1, \dots, p$ ) have the same meanings as in Eqs. (2.14.5) and (2.13.1), respectively, and  $\tilde{F}_r$  ( $r = 1, \dots, p$ ) are defined in Eqs. (4.4.1). If  $u_1, \dots, u_n$  are defined as  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ), then Eqs. (1) give way to

$$\frac{\partial K}{\partial q_r} + \sum_{s=p+1}^n \frac{\partial K}{\partial q_s} C_{sr} + \tilde{F}_r = 0 \quad (r = 1, \dots, p) \quad (2)$$

where  $C_{sr}$  ( $s = p + 1, \dots, n; r = 1, \dots, p$ ) has the same meaning as in Eq. (5.1.13).

If  $S$  is a holonomic system with  $u_r$  defined as in Eqs. (2.12.1), so that Eqs. (2.14.5) apply, then

$$\sum_{s=1}^n \frac{\partial K}{\partial q_s} W_{sr} + F_r = 0 \quad (r = 1, \dots, n) \quad (3)$$

during motions resembling states of rest; and if  $u_r = \dot{q}_r$  ( $r = 1, \dots, n$ ), then Eqs. (3) are replaced with

$$\frac{\partial K}{\partial q_r} + F_r = 0 \quad (r = 1, \dots, n) \quad (4)$$

When using Eqs. (1)–(4), one can work with expressions for  $K$ ,  $\tilde{F}_r$  ( $r = 1, \dots, p$ ), or  $F_r$  ( $r = 1, \dots, n$ ) that apply only when  $\dot{q}_1 = \dots = \dot{q}_n = 0$ , rather than during the most general motion of  $S$ ; generally, this simplifies matters considerably.

If  $S$  possesses a potential energy  $V$  in  $N$ , and  $\mathcal{L}$  is defined as

$$\mathcal{L} \triangleq K - V \quad (5)$$

then Eqs. (1)–(4) can be replaced with

$$\sum_{s=1}^n \frac{\partial \mathcal{L}}{\partial q_s} \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) = 0 \quad (r = 1, \dots, p) \quad (6)$$

$$\frac{\partial \mathcal{L}}{\partial q_r} + \sum_{s=p+1}^n \frac{\partial \mathcal{L}}{\partial q_s} C_{sr} = 0 \quad (r = 1, \dots, p) \quad (7)$$

$$\sum_{s=1}^n \frac{\partial \mathcal{L}}{\partial q_s} W_{sr} = 0 \quad (r = 1, \dots, n) \quad (8)$$

$$\frac{\partial \mathcal{L}}{\partial q_r} = 0 \quad (r = 1, \dots, n) \quad (9)$$

respectively. Use of these equations obviates forming expressions for accelerations.

**Derivations** If  $K$ , when regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , does not involve  $t$  explicitly, then the partial derivatives  $\partial K / \partial \dot{q}_r$  ( $r = 1, \dots, n$ ) are functions  $f_r$  ( $r = 1, \dots, n$ ) of precisely the  $2n$  variables  $q_r$  and  $\dot{q}_r$  ( $r = 1, \dots, n$ ), so that one can write

$$\frac{\partial K}{\partial \dot{q}_r} = f_r(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \quad (r = 1, \dots, n) \quad (10)$$

and

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_r} \right) = \sum_{s=1}^n \left( \frac{\partial f_r}{\partial q_s} \dot{q}_s + \frac{\partial f_r}{\partial \dot{q}_s} \ddot{q}_s \right) \quad (r = 1, \dots, n) \quad (11)$$

But, since  $q_s = \bar{q}_s$  ( $s = 1, \dots, n$ ) by hypothesis,

$$\dot{q}_s = \ddot{q}_s = 0 \quad (s = 1, \dots, n) \quad (12)$$

so that

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_r} \right)_{(11,12)} = 0 \quad (r = 1, \dots, n) \quad (13)$$

and use of these relationships in conjunction with the four equations in Problem 11.12 (taken in reverse order) leads directly to Eqs. (1)–(4).

When  $S$  possesses a potential energy  $V$  in  $N$ ,  $\tilde{F}_r$  in Eqs. (1) can be replaced with the right-hand member of Eq. (5.1.18), which shows that

$$\sum_{s=1}^n \left( \frac{\partial K}{\partial q_s} - \frac{\partial V}{\partial q_s} \right) \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) = 0 \quad (r = 1, \dots, p) \quad (14)$$

under these circumstances. Now,

$$\frac{\partial K}{\partial q_s} - \frac{\partial V}{\partial q_s} = \frac{\partial}{\partial q_s} (K - V) = \frac{\partial \mathcal{L}}{\partial q_s} \quad (s = 1, \dots, n) \quad (15)$$

Substitution from Eqs. (15) into Eqs. (14) produces Eqs. (6). Similarly, Eqs. (2), (15), and (5.1.14) lead to Eqs. (7); Eqs. (3), (15), and (5.1.9) underlie Eqs. (8); and Eqs. (4), (15), and (5.1.2) can be used to establish the validity of Eqs. (9).

**Example** Two uniform bars,  $A$  and  $B$ , each of mass  $m$  and length  $L$ , are connected by a pin, and  $A$  is pinned to a vertical shaft that is made to rotate with constant angular speed  $\Omega$ , as indicated in Fig. 6.7.1. (The axes of the pins are

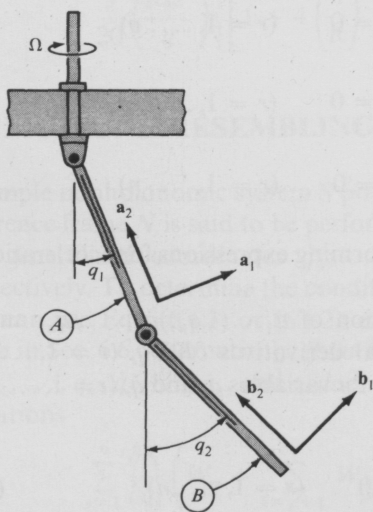


Figure 6.7.1

parallel to each other and horizontal.) This system can move in such a way that  $q_1$  and  $q_2$  (see Fig. 6.7.1) remain constant. Equations (9) will be used to formulate equations governing  $q_1$  and  $q_2$  during such motions.

When  $\dot{q}_1 = \dot{q}_2 = 0$ , the angular velocities of  $A$  and  $B$  can be written

$$\omega^A = -\Omega(s_1 \mathbf{a}_1 + c_1 \mathbf{a}_2) \quad \omega^B = -\Omega(s_2 \mathbf{b}_1 + c_2 \mathbf{b}_2) \quad (16)$$

where  $\mathbf{a}_i$  and  $\mathbf{b}_i$  ( $i = 1, 2$ ) are unit vectors directed as shown in Fig. 6.7.1; the velocity of  $B^*$ , the mass center of  $B$ , is

$$\mathbf{v}^{B^*} = L\Omega \left( s_1 + \frac{s_2}{2} \right) \mathbf{b}_3 \quad (17)$$

and the kinetic energy  $K$  is thus given by

$$\begin{aligned} K &= \frac{1}{2} \frac{mL^2}{3} \Omega^2 s_1^2 + \frac{1}{2} \frac{mL^2}{12} \Omega^2 s_2^2 + \frac{1}{2} mL^2 \Omega^2 \left( s_1 + \frac{s_2}{2} \right)^2 \\ &\quad \text{(Problem 10.4, 16)} \quad \text{(5.4.2)} \quad \text{(5.4.7, 16)} \quad \text{(5.4.4, 17)} \\ &= \frac{mL^2 \Omega^2}{2} \left( \frac{4}{3} s_1^2 + s_1 s_2 + \frac{1}{3} s_2^2 \right) \end{aligned} \quad (18)$$

The potential energy  $V$  can be expressed as

$$V = -mgL \left( \frac{1}{2} c_1 + c_1 + \frac{1}{2} c_2 \right) = -\frac{mgL}{2} (3c_1 + c_2) \quad (19)$$

Consequently,

$$\mathcal{L} = \frac{mL^2 \Omega^2}{2} \left( \frac{4}{3} s_1^2 + s_1 s_2 + \frac{1}{3} s_2^2 \right) + \frac{mgL}{2} (3c_1 + c_2) \quad (20)$$

(5) (18) (19)

and, setting  $\partial \mathcal{L} / \partial q_r = 0$  ( $r = 1, 2$ ), one obtains the desired equations, namely,

$$\left( \frac{L\Omega^2}{g} \right) c_1 (8s_1 + 3s_2) - 9s_1 = 0 \quad (21)$$

and

$$\left( \frac{L\Omega^2}{g} \right) c_2 (3s_1 + 2s_2) - 3s_2 = 0 \quad (22)$$

## EXTRACTION OF INFORMATION FROM EQUATIONS OF MOTION

This chapter is intended to bring to fruition the effort that has been expended by the reader in learning the material covered in the first six chapters. Following the introduction, in Sec. 7.1, of terminology employed in connection with solutions of equations of motion, several theorems that permit the construction of integrals of equations of motion are presented in Secs. 7.2 and 7.3. Exact closed-form solutions are considered in Sec. 7.4, and means for obtaining numerical results when closed-form solutions are unavailable are discussed in Sec. 7.5. These techniques, used in conjunction with material in Sec. 6.3, permit the evaluation of constraint forces and constraint torques, as is shown in Sec. 7.6. Section 7.7 deals with a purely mathematical problem that arises frequently in dynamics, namely, finding real solutions of a set of nonlinear, nondifferential equations. The concepts of generalized impulse and generalized momentum are introduced in Sec. 7.8 in preparation for the presentation, in Sec. 7.9, of a method for analyzing phenomena involving collisions. Finally, Sec. 7.10 contains an exposition of the principal concepts underlying the theory of small vibrations of mechanical systems.

### 7.1 INTEGRALS OF EQUATIONS OF MOTION

The behavior of a simple nonholonomic system  $S$  possessing  $p$  degrees of freedom in a Newtonian reference frame  $N$  (see Sec. 2.13) is governed by  $2n$  equations called, collectively, the *equations of motion* of  $S$  in  $N$ . This set of equations consists of

three subsets: the  $n$  *kinematical differential equations*, Eqs. (2.12.1) or, equivalently, Eqs. (2.14.5); the  $n - p$  *nonholonomic constraint equations*, Eqs. (2.13.1); and the  $p$  *dynamical differential equations*, Eqs. (6.1.1). If  $S$  is a holonomic system, then  $n = p$ , the nonholonomic constraint equations are absent, and Eqs. (6.1.2) furnish  $n$  dynamical equations.

A solution of the equations of motion is said to be in hand when the generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10) and generalized speeds  $u_1, \dots, u_n$  (see Sec. 2.12) are known as functions of the time  $t$ . The *general solution* of the equations of motion consists of  $2n$  independent equations of the form

$$f_i(q_1, \dots, q_n, u_1, \dots, u_n, t) = C_i \quad (i = 1, \dots, 2n) \quad (1)$$

where  $f_i$  ( $i = 1, \dots, 2n$ ) is a function whose total time-derivative vanishes, whenever  $q_1, \dots, q_n$  and  $u_1, \dots, u_n$  satisfy all equations of motion, and  $C_1, \dots, C_{2n}$  are arbitrary constants. An equation having the form of Eqs. (1) is called an *integral of the equations of motion*. Thus, the general solution of the equations of motion consists of  $2n$  integrals.

**Examples** For the nonholonomic system  $S$  formed by the particle  $P_1$  and the sharp-edged circular disk  $D$  connected by a rigid rod  $R$  as in the example in Sec. 2.13,  $p = 2$  and  $n = 3$ . When the kinematical differential equations are taken to be [see Eqs. (2.12.7)]

$$u_1 = \dot{q}_1 c_3 + \dot{q}_2 s_3 \quad u_2 = -\dot{q}_1 s_3 + \dot{q}_2 c_3 \quad u_3 = \dot{q}_3 \quad (2)$$

then the nonholonomic constraint equation expressing the restriction that the center of  $D$  may not move perpendicularly to  $R$  is [see Eq. (2.13.12)]

$$u_3 = -\frac{u_2}{L} \quad (3)$$

Furthermore, if line  $Y$  in Fig. 2.13.1 is vertical,  $P_1$  and  $D$  have masses  $m_1$  and  $m_2$ , respectively, and  $P_1$  is subjected to the action of a contact force  $\mathbf{K}$  given by Eq. (5.1.26), then the generalized active forces for  $S$  are given by [see Eqs. (5.1.39)]

$$\tilde{F}_1 = k - (m_1 + m_2)gs_3 \quad (4)$$

$$\tilde{F}_2 = \frac{k}{L}(q_1 s_3 - q_2 c_3) - m_1 g c_3 \quad (5)$$

while the generalized inertia forces can be written (see Problem 8.16)

$$\tilde{F}_1^* = (m_1 + m_2)(\omega^2 q_1 c_3 - \dot{u}_1) - \frac{m_1 u_2^2}{L} + m_2 L \omega^2 c_3^2 \quad (6)$$

$$\tilde{F}_2^* = -m_1 \left( \dot{u}_2 + \omega^2 q_1 s_3 - \frac{u_1 u_2}{L} \right) \quad (7)$$

Substitution from Eqs. (4)–(7) into Eqs. (6.1.1) thus yields the dynamical differential equations

$$k - (m_1 + m_2)gs_3 + (m_1 + m_2)(\omega^2 q_1 c_3 - \dot{u}_1) - \frac{m_1 u_2^2}{L} + m_2 L \omega^2 c_3^2 = 0 \quad (8)$$

and

$$\frac{k}{L}(q_1 s_3 - q_2 c_3) - m_1 g c_3 - m_1 \left( \dot{u}_2 + \omega^2 q_1 s_3 - \frac{u_1 u_2}{L} \right) = 0 \quad (9)$$

Equations (2), (3), (8), and (9) are the six equations of motion of  $S$ .

As will be shown presently, the equation

$$-k(q_1 c_3 + q_2 s_3) + g[(m_1 + m_2)q_2 + m_2 L s_3] + \frac{1}{2}[(m_1 + m_2)u_1^2 + m_1 u_2^2] - \frac{1}{2}\omega^2[m_1 q_1^2 + m_2(q_1 + L c_3)^2] = C \quad (10)$$

where  $C$  is an arbitrary constant, is an integral of the equations of motion of  $S$ . It does not matter for present purposes how this integral was found; this subject is discussed in its own right in the example in Sec. 7.2.

Denoting the left-hand member of Eq. (10) by  $f(q_1, q_2, q_3, u_1, u_2, u_3, t)$ , and expressing the total time-derivative of  $f$  as

$$\dot{f} = \sum_{r=1}^3 \left( \frac{\partial f}{\partial q_r} \dot{q}_r + \frac{\partial f}{\partial u_r} \dot{u}_r \right) + \frac{\partial f}{\partial t} \quad (11)$$

one finds that

$$\begin{aligned} \dot{f}_{(11,10)} = & \{-kc_3 - \omega^2[m_1 q_1 + m_2(q_1 + L c_3)]\}\dot{q}_1 \\ & + [-ks_3 + g(m_1 + m_2)]\dot{q}_2 \\ & + [k(q_1 s_3 - q_2 c_3) + g m_2 L c_3 + \omega^2 m_2(q_1 + L c_3)L s_3]\dot{q}_3 \\ & + (m_1 + m_2)u_1 \dot{u}_1 + m_1 u_2 \dot{u}_2 \end{aligned} \quad (12)$$

Solution of the kinematical differential equations, Eqs. (2), for  $\dot{q}_1$ ,  $\dot{q}_2$ , and  $\dot{q}_3$  yields

$$\dot{q}_1 = u_1 c_3 - u_2 s_3 \quad \dot{q}_2 = u_1 s_3 + u_2 c_3 \quad \dot{q}_3 = u_3 \quad (13)$$

while solution of the dynamical differential equations for  $\dot{u}_1$  and  $\dot{u}_2$  leads to

$$\dot{u}_1 = \omega^2 c_3 \left( q_1 + \frac{m_2 L c_3}{m_1 + m_2} \right) + \frac{k}{m_1 + m_2} - g s_3 - \frac{m_1 u_2^2}{(m_1 + m_2)L} \quad (14)$$

$$\dot{u}_2 = -\omega^2 q_1 s_3 + \frac{k}{m_1 L}(q_1 s_3 - q_2 c_3) - g c_3 + \frac{u_1 u_2}{L} \quad (15)$$



If  $u_3$  in the third of Eqs. (13) now is replaced with  $-u_2/L$  in accordance with the nonholonomic constraint equation, Eq. (3), and Eqs. (13)–(15) then are used to eliminate  $\dot{q}_1$ ,  $\dot{q}_2$ ,  $\dot{q}_3$ ,  $\dot{u}_1$ , and  $\dot{u}_2$  from Eq. (12),  $\dot{f}$  is found to be equal to zero, and this establishes Eq. (10) as an integral of the equations of motion of  $S$ . (To obtain the general solution of the equations of motion, one must produce five more integrals.)

As a second example, we consider a system governed by differential equations whose explicit solution is well known, namely, the *harmonic oscillator*, that is, any system whose behavior is governed by the differential equation

$$\ddot{q} + \omega^2 q = 0 \quad (16)$$

where  $\omega^2$  is a constant. This equation can be replaced with the two first-order equations

$$\dot{q} = u \quad \dot{u} = -\omega^2 q \quad (17)$$

the first of which is simply a definition of  $u$  and plays the role of a kinematical differential equation, while the second is a dynamical differential equation. Thus,  $n = p = 1$ .

Using the same procedure as in the preceding example, one can verify that

$$q \sin \omega t + \frac{u}{\omega} \cos \omega t = C_1 \quad (18)$$

and

$$q \cos \omega t - \frac{u}{\omega} \sin \omega t = C_2 \quad (19)$$

are integrals of the equations of motion, Eqs. (17). Together, these two integrals constitute the general solution of Eqs. (17). To bring this solution into a form that may be more familiar, one can solve Eq. (18) for  $u/\omega$ , substitute the result into Eq. (19), and obtain, after simplification,

$$q = C_1 \sin \omega t + C_2 \cos \omega t \quad (20)$$

## 7.2 THE ENERGY INTEGRAL

When a system  $S$  possesses a potential energy  $V$  in a Newtonian reference frame  $N$  (see Sec. 5.1), there may exist an integral of the equations of motion of  $S$  in  $N$  (see Sec. 7.1) that can be expressed as

$$H = C \quad (1)$$

where  $C$  is a constant and  $H$ , called the *Hamiltonian* of  $S$  in  $N$ , is defined as

$$H \triangleq V + K_2 - K_0 \quad (2)$$

with  $K_0$  and  $K_2$  given by Eqs. (5.5.7) and (5.5.9), respectively. Equation (1) is called the *energy integral* of the equations of motion of  $S$  in  $N$ . The conditions

**Table 7.2.1**

	$u_r = \dot{q}_r$ $X_r = 0, W_{rs} = \delta_{rs}$ $(r, s = 1, \dots, n)$	$u_r = \sum_{s=1}^n Y_{rs} \dot{q}_s + Z_r$ $(r = 1, \dots, n)$
Holonomic	Eq. (5.1.3) and [Eq. (5.6.1) or Eq. (5.6.3)]	Eq. (5.1.10) and [Eq. (5.6.1) or Eqs. (5.6.3)–(5.6.5)]
Nonholonomic	Eq. (5.1.15) and [Eq. (5.6.1) or Eqs. (5.6.3), (5.6.4)]	Eq. (5.1.19) and [Eq. (5.6.1) or Eqs. (5.6.3)–(5.6.5)]

under which it exists depend on the manner in which generalized speeds are introduced and on whether  $S$  is a holonomic or nonholonomic system. In Table 7.2.1, sufficient conditions for the existence of the energy integral of  $S$  in  $N$  are indicated by listing the numbers of equations satisfaction of which ensures the existence of an energy integral.

If the kinetic energy of  $S$  is a homogeneous quadratic function of  $u_1, \dots, u_n$ , that is, if  $K_0 = K_1 = 0$ , so that  $K = K_2$  [see Eq. (5.5.1)], then Eqs. (1) and (2) yield

$$V + K = E \quad (3)$$

where  $E$  is a constant called the *mechanical energy* of  $S$  in  $N$ . Equation (3) expresses the *principle of conservation of mechanical energy*.

**Derivation** Suppose that Eqs. (5.1.19) and [(5.6.1) or (5.6.3)–(5.6.5)] are satisfied (see Table 7.2.1). Then

$$\dot{V} \underset{(5.1.20)}{=} - \sum_{r=1}^p \tilde{F}_r u_r \quad (4)$$

and

$$\dot{K}_2 - \dot{K}_0 \underset{(5.6.2)}{=} - \sum_{r=1}^p \tilde{F}_r^* u_r \quad (5)$$

so that

$$\dot{H} \underset{(2)}{=} \dot{V} + \dot{K}_2 - \dot{K}_0 \underset{(4,5)}{=} - \sum_{r=1}^p (\tilde{F}_r + \tilde{F}_r^*) u_r \underset{(6.1.1)}{=} 0 \quad (6)$$

and Eq. (1) follows immediately. The derivations of Eq. (1) for the remaining three cases proceed similarly when the remaining entries in Table 7.2.1 are taken into account.

**Examples** A potential energy  $V$  and the kinetic energy functions  $K_0$  and  $K_2$  for the system  $S$  considered in the first example in Sec. 7.1 are given by

$$V = -k(q_1c_3 + q_2s_3) + g[(m_1 + m_2)q_2 + m_2Ls_3] \quad (7)$$

(5.1.64)

$$K_0 = \frac{1}{2}\omega^2[m_1q_1^2 + m_2(q_1 + Lc_3)^2] \quad (8)$$

(5.5.11)

and

$$K_2 = \frac{1}{2}[(m_1 + m_2)u_1^2 + m_1u_2^2] \quad (9)$$

(5.5.12)

Setting  $V + K_2 - K_0$  equal to an arbitrary constant, one arrives at Eq. (7.1.10). In the example in Sec. 7.1, the equations of motion were brought into play to show that Eq. (7.1.10) is an integral of these equations. Alternatively, one can establish this fact with the aid of Eqs. (5.1.19) and [(5.6.1) or (5.6.3)–(5.6.5)], all of which are satisfied.

For the system depicted in Fig. 4.4.1, a potential energy  $V$  is given by

$$V = -[m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)]g \cos \theta$$

(5.2.25)

$$+ \frac{1}{2}[k_1q_1^2 + k_2(q_2 - q_1)^2] \quad (10)$$

and the kinetic energy  $K$  can be written

$$K = \frac{1}{2}\{m_1[u_1\mathbf{t}_1 + (L_1 + q_1)\dot{\theta}\mathbf{t}_2]^2 + m_2[u_2\mathbf{t}_1 + (L_1 + L_2 + q_2)\dot{\theta}\mathbf{t}_2]^2\}$$

(4.4.6) (4.4.7)

$$= \frac{1}{2}\{m_1u_1^2 + m_2u_2^2 + [m_1(L + q_1)^2 + m_2(L_1 + L_2 + q_2)^2]\dot{\theta}^2\} \quad (11)$$

so that

$$K_0 = \frac{1}{2}[m_1(L + q_1)^2 + m_2(L_1 + L_2 + q_2)^2]\dot{\theta}^2 \quad (12)$$

(11)

and

$$K_2 = \frac{1}{2}(m_1u_1^2 + m_2u_2^2) \quad (13)$$

(11)

Since the system is holonomic and, in accordance with Eqs. (4.4.5),  $u_r = \dot{q}_r$  ( $r = 1, 2$ ), Eq. (1) furnishes an integral of the equations of motion if (see Table 7.2.1) Eqs. (5.1.3) and (5.6.3) are satisfied, that is, if [see Eq. (5.1.3)]

$$\frac{\partial V}{\partial t} = [m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)]g \sin \theta \dot{\theta} = 0 \quad (14)$$

(10)

and [see Eq. (5.6.3)]

$$\frac{\partial K}{\partial t} = [m_1(L + q_1)^2 + m_2(L_1 + L_2 + q_2)^2]\dot{\theta}\ddot{\theta} = 0 \quad (15)$$

(11)

Clearly, both of these requirements are fulfilled if  $\dot{\theta} \equiv 0$ , so that

$$\theta = \bar{\theta} \quad (16)$$

where  $\bar{\theta}$  is a constant. Moreover,  $K$  [see Eq. (11)] then is a homogeneous quadratic function of  $u_1$  and  $u_2$ , and Eq. (3) applies, which makes it possible to write [see Eqs. (10) and (11)]

$$- [m_1(L_1 + q_1) + m_2(L_1 + L_2 + q_2)]g \cos \bar{\theta} + \frac{1}{2}[k_1 q_1^2 + k_2(q_2 - q_1)^2 + m_1 u_1^2 + m_2 u_2^2] = E \quad (17)$$

with  $E$  an arbitrary constant.

### 7.3 MOMENTUM INTEGRALS

If  $\sigma$  is the set of all contact and distance forces acting on the particles of a system  $S$ ,  $\mathbf{R}$  is the resultant of  $\sigma$  (see Sec. 4.1),  $\mathbf{L}$  is the linear momentum of  $S$  in a Newtonian reference frame  $N$ ,  $\mathbf{n}$  is a unit vector, and

$$\mathbf{R} \cdot \mathbf{n} + \mathbf{L} \cdot \frac{^N d\mathbf{n}}{dt} = 0 \quad (1)$$

then the equation

$$\mathbf{L} \cdot \mathbf{n} = C \quad (2)$$

where  $C$  is any constant, is an integral of the equations of motion of  $S$  in  $N$ . Similarly, if  $\mathbf{M}$  is the moment of  $\sigma$  about  $S^*$ , the mass center of  $S$ , or about a point  $O$  fixed in  $N$ ,  $\mathbf{H}$  is the angular momentum of  $S$  about  $S^*$  or  $O$  in  $N$ , and

$$\mathbf{M} \cdot \mathbf{n} + \mathbf{H} \cdot \frac{^N d\mathbf{n}}{dt} = 0 \quad (3)$$

then the equation

$$\mathbf{H} \cdot \mathbf{n} = C \quad (4)$$

where  $C$  is any constant, is an integral of the equations of motion of  $S$  in  $N$ . Finally, if  $S$  is a holonomic system possessing generalized coordinates  $q_1, \dots, q_n$  and a kinetic potential  $\mathcal{L}$  in  $N$  (see Problem 11.13), and  $q_k$  is absent from  $\mathcal{L}$  when  $\mathcal{L}$  is expressed as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ , then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = C \quad (5)$$

where  $C$  is any constant, is an integral of the equations of motion of  $S$  in  $N$ .

Equations (2), (4), and (5) are called *momentum integrals* of the equations of motion, and  $q_k$  is referred to as a *cyclic* or *ignorable* coordinate.

**Derivations** In accordance with the *linear momentum principle*,

$$\frac{{}^N d\mathbf{L}}{dt} = \mathbf{R} \quad (6)$$

Hence,

$$\frac{d}{dt}(\mathbf{L} \cdot \mathbf{n}) = \frac{{}^N d\mathbf{L}}{dt} \cdot \mathbf{n} + \mathbf{L} \cdot \frac{{}^N d\mathbf{n}}{dt} = \underset{(6)}{\mathbf{R} \cdot \mathbf{n}} + \mathbf{L} \cdot \frac{{}^N d\mathbf{n}}{dt} \quad (7)$$

and, when Eq. (1) is satisfied, then Eq. (7) yields

$$\frac{d}{dt}(\mathbf{L} \cdot \mathbf{n}) = 0 \quad (8)$$

from which Eq. (2) follows immediately. Similarly, the *angular momentum principle* asserts that

$$\frac{{}^N d\mathbf{H}}{dt} = \mathbf{M} \quad (9)$$

so that

$$\frac{d}{dt}(\mathbf{H} \cdot \mathbf{n}) = \frac{{}^N d\mathbf{H}}{dt} \cdot \mathbf{n} + \mathbf{H} \cdot \frac{{}^N d\mathbf{n}}{dt} = \underset{(9)}{\mathbf{M} \cdot \mathbf{n}} + \mathbf{H} \cdot \frac{{}^N d\mathbf{n}}{dt} \quad (10)$$

and this equation together with Eq. (3) leads to Eq. (4). Finally, when  $q_k$  is absent from  $\mathcal{L}$ , Eq. (5) is an immediate consequence of Lagrange's equations of the second kind (see Problem 11.13).

**Examples** Figure 7.3.1 is a schematic representation of a *gyrostat*  $G$  consisting of a rigid body  $A$  and an axially symmetric rotor  $B$  that is constrained to rotate relative to  $A$  about an axis fixed in  $A$ .  $G$  has a mass  $M$ , and  $\mathbf{I}_G$ , the central

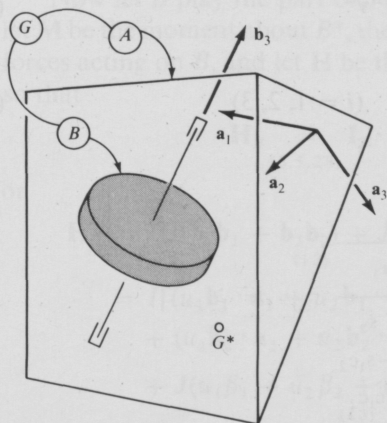


Figure 7.3.1

inertia dyadic of  $G$ , expressed in terms of  $K_1$ ,  $K_2$ , and  $K_3$ , the central principal moments of inertia of  $G$ , is given by

$$\mathbf{I}_G = K_1 \mathbf{a}_1 \mathbf{a}_1 + K_2 \mathbf{a}_2 \mathbf{a}_2 + K_3 \mathbf{a}_3 \mathbf{a}_3 \quad (11)$$

where  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  are mutually perpendicular unit vectors fixed in  $A$  and parallel to central principal axes of  $G$ . The axis of symmetry of  $B$  is parallel to a unit vector  $\mathbf{b}_3$ , and  $\mathbf{I}_B$ , the central inertia dyadic of  $B$ , can be expressed as

$$\mathbf{I}_B = I(\mathbf{b}_1 \mathbf{b}_1 + \mathbf{b}_2 \mathbf{b}_2) + J \mathbf{b}_3 \mathbf{b}_3 \quad (12)$$

where  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are unit vectors fixed in  $A$ , perpendicular to each other, and such that  $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ .

When  $G$  moves in a Newtonian reference frame  $N$  in the absence of external constraints, it is convenient to work with generalized speeds  $u_1, \dots, u_7$  defined as

$$u_i \triangleq {}^N \boldsymbol{\omega}^A \cdot \mathbf{a}_i \quad (i = 1, 2, 3) \quad (13)$$

$$u_4 \triangleq {}^A \boldsymbol{\omega}^B \cdot \mathbf{b}_3 \quad (14)$$

$$u_{4+i} \triangleq {}^N \mathbf{v}^{G^*} \cdot \mathbf{n}_i \quad (i = 1, 2, 3) \quad (15)$$

where  $G^*$  is the mass center of  $G$  and  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ ,  $\mathbf{n}_3$  are mutually perpendicular unit vectors fixed in  $N$ . As generalized coordinates one can use, for example, angles  $q_1$ ,  $q_2$ , and  $q_3$  like those in Problem 1.1 to characterize the relative orientations of  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ ,  $\mathbf{n}_3$  and  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$ ; an angle  $q_4$  between two lines that are perpendicular to  $\mathbf{b}_3$ , one line fixed in  $A$ , the other fixed in  $B$ ; and Cartesian coordinates  $q_5$ ,  $q_6$ , and  $q_7$  of  $G^*$  relative to a set of axes fixed in  $N$ . Table 7.3.1 shows the relationship between  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  and  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ ,  $\mathbf{n}_3$ .

Letting  $G$  play the role of  $S$ , suppose that  $G$  moves in  $N$  in the absence of external forces. Then  $\mathbf{R}$  in Eq. (1) is equal to zero and, if one takes for  $\mathbf{n}$  in Eq. (1) any one of  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ ,  $\mathbf{n}_3$ , then Eq. (1) is satisfied because  ${}^N d\mathbf{n}_i/dt = 0$  ( $i = 1, 2, 3$ ). Since  $\mathbf{L}$ , the linear momentum of  $G$  in  $N$ , is given by

$$\mathbf{L} = M {}^N \mathbf{v}^{G^*} \quad (16)$$

one can, therefore, write

$${}^N \mathbf{v}^{G^*} \cdot \mathbf{n}_i = C_i \quad (i = 1, 2, 3) \quad (17)$$

(2)

**Table 7.3.1**

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{n}_1$	$c_2 c_3$	$-c_2 s_3$	$s_2$
$\mathbf{n}_2$	$s_1 s_2 c_3 + s_3 c_1$	$-s_1 s_2 s_3 + c_3 c_1$	$-s_1 c_2$
$\mathbf{n}_3$	$-c_1 s_2 c_3 + s_3 s_1$	$c_1 s_2 s_3 + c_3 s_1$	$c_1 c_2$

which leads with the aid of Eqs. (15) to the three integrals

$$u_{4+i} = C_i \quad (i = 1, 2, 3) \quad (18)$$

In the absence of external forces,  $\mathbf{M}$  in Eq. (3) is equal to zero and, with  $\mathbf{n}$  equal to  $\mathbf{n}_i$  ( $i = 1, 2, 3$ ), Eq. (3) is satisfied. Consequently,

$$\mathbf{H}_G \cdot \mathbf{n}_i = C_{3+i} \quad (i = 1, 2, 3) \quad (19)$$

where  $\mathbf{H}_G$ , the central angular momentum of  $G$  in  $N$ , can be written (see Problem 6.5)

$$\mathbf{H}_G = \mathbf{I}_G \cdot {}^N\boldsymbol{\omega}^A + \mathbf{I}_B \cdot {}^A\boldsymbol{\omega}^B \quad (20)$$

or

$$\mathbf{H}_G = K_1 u_1 \mathbf{a}_1 + K_2 u_2 \mathbf{a}_2 + K_3 u_3 \mathbf{a}_3 + Ju_4 \mathbf{b}_3 \quad (21)$$

(20)                      (11, 13)                      (12, 14)

Consequently,

$$K_1 u_1 \mathbf{a}_1 \cdot \mathbf{n}_i + K_2 u_2 \mathbf{a}_2 \cdot \mathbf{n}_i + K_3 u_3 \mathbf{a}_3 \cdot \mathbf{n}_i + Ju_4 \mathbf{b}_3 \cdot \mathbf{n}_i = C_{3+i} \quad (i = 1, 2, 3) \quad (22)$$

(19, 21)

Furthermore,  $\mathbf{b}_3$  can be written

$$\mathbf{b}_3 = \beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_2 + \beta_3 \mathbf{a}_3 \quad (23)$$

where  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  are constants. Referring to Table 7.3.1, one thus arrives at the integrals

$$(K_1 u_1 + Ju_4 \beta_1) c_2 c_3 - (K_2 u_2 + Ju_4 \beta_2) c_2 s_3 + (K_3 u_3 + Ju_4 \beta_3) s_2 = C_4 \quad (24)$$

$$(K_1 u_1 + Ju_4 \beta_1) (s_1 s_2 c_3 + s_3 c_1) + (K_2 u_2 + Ju_4 \beta_2) (-s_1 s_2 s_3 + c_3 c_1) - (K_3 u_3 + Ju_4 \beta_3) s_1 c_2 = C_5 \quad (25)$$

$$(K_1 u_1 + Ju_4 \beta_1) (-c_1 s_2 c_3 + s_3 s_1) + (K_2 u_2 + Ju_4 \beta_2) (c_1 s_2 s_3 + c_3 s_1) + (K_3 u_3 + Ju_4 \beta_3) c_1 c_2 = C_6 \quad (26)$$

Now let  $B$  play the part of  $S$  in connection with Eqs. (3) and (4); that is, let  $\mathbf{M}$  be the moment about  $B^*$ , the mass center of  $B$ , of all contact and distance forces acting on  $B$ , and let  $\mathbf{H}$  be the central angular momentum  $\mathbf{H}_B$  of  $B$  in  $N$  so that

$$\mathbf{H}_B = \mathbf{I}_B \cdot {}^N\boldsymbol{\omega}^B = \mathbf{I}_B \cdot ({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B) \quad (27)$$

(3.5, 2.8)                      (2.4, 1)

or

$$\begin{aligned} \mathbf{H}_B &= [(I(\mathbf{b}_1 \mathbf{b}_1 + \mathbf{b}_2 \mathbf{b}_2) + J\mathbf{b}_3 \mathbf{b}_3) \cdot (u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + u_3 \mathbf{a}_3 + u_4 \mathbf{b}_3)] \\ &= I[(u_1 \mathbf{b}_1 \cdot \mathbf{a}_1 + u_2 \mathbf{b}_1 \cdot \mathbf{a}_2 + u_3 \mathbf{b}_1 \cdot \mathbf{a}_3) \mathbf{b}_1 \\ &\quad + (u_1 \mathbf{b}_2 \cdot \mathbf{a}_1 + u_2 \mathbf{b}_2 \cdot \mathbf{a}_2 + u_3 \mathbf{b}_2 \cdot \mathbf{a}_3) \mathbf{b}_2] \\ &\quad + J(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3 + u_4) \mathbf{b}_3 \end{aligned} \quad (28)$$

(27)                      (12)                      (13)                      (14)

Finally, let  $\mathbf{b}_3$  play the role of  $\mathbf{n}$  in Eq. (3), and suppose that  $B$  is completely free to rotate relative to  $A$ , so that

$$\mathbf{M} \cdot \mathbf{b}_3 = 0 \quad (29)$$

Then, since

$$\frac{{}^N d\mathbf{b}_3}{dt} \underset{(2.1.2)}{=} {}^N \boldsymbol{\omega}^A \times \mathbf{b}_3 \underset{(13)}{=} u_1 \mathbf{a}_1 \times \mathbf{b}_3 + u_2 \mathbf{a}_2 \times \mathbf{b}_3 + u_3 \mathbf{a}_3 \times \mathbf{b}_3 \quad (30)$$

one can use the relationships

$$\mathbf{b}_1 \cdot \mathbf{a}_1 \times \mathbf{b}_3 = \mathbf{b}_3 \times \mathbf{b}_1 \cdot \mathbf{a}_1 = \mathbf{b}_2 \cdot \mathbf{a}_1 \quad (31)$$

$$\mathbf{b}_2 \cdot \mathbf{a}_1 \times \mathbf{b}_3 = \mathbf{b}_3 \times \mathbf{b}_2 \cdot \mathbf{a}_1 = -\mathbf{b}_1 \cdot \mathbf{a}_1 \quad (32)$$

and so forth, to verify that

$$\mathbf{H}_B \cdot \frac{{}^N d\mathbf{b}_3}{dt} \underset{(28,30)}{=} 0 \quad (33)$$

Consequently, Eq. (3) is satisfied, and Eqs. (4) and (28) lead to the integral

$$u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3 + u_4 = C \quad (34)$$

The same integral can be found by noting that the Lagrangian  $\mathcal{L}$  is here equal to the kinetic energy of  $G$  in  $N$ , so that (see Problem 10.8)

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(K_1 u_1^2 + K_2 u_2^2 + K_3 u_3^2) + J u_4 \left( u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3 + \frac{u_4}{2} \right) \\ & + \frac{1}{2} M(u_5^2 + u_6^2 + u_7^2) \end{aligned} \quad (35)$$

and  $q_4$  is absent from  $\mathcal{L}$  when  $u_1, \dots, u_7$  are expressed in terms of  $q_1, \dots, q_7$  and  $\dot{q}_1, \dots, \dot{q}_7$ ; however,  $\dot{q}_4$  appears in Eq. (35) because  $u_4 = \dot{q}_4$ . Consequently,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_4} = \frac{\partial \mathcal{L}}{\partial u_4} \underset{(35)}{=} J(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3 + u_4) \quad (36)$$

and, setting the right-hand member of this equation equal to a constant in accordance with Eq. (5), one recovers Eq. (34).

The integrals reported in Eqs. (18) and (24) also can be obtained by using Eq. (5) in conjunction with Eq. (35), but those expressed in Eqs. (25) and (26) cannot, for  $q_2$  and  $q_3$  appear in  $\mathcal{L}$  when  $u_1, u_2$ , and  $u_3$  are replaced in Eq. (35) with

$$u_1 = \dot{q}_1 c_2 c_3 + \dot{q}_2 s_3 \quad (37)$$

$$u_2 = -\dot{q}_1 c_2 s_3 + \dot{q}_2 c_3 \quad (38)$$

$$u_3 = \dot{q}_1 s_2 + \dot{q}_3 \quad (39)$$

respectively.



## 7.4 EXACT CLOSED-FORM SOLUTIONS

Usually, it is difficult to find in closed form the exact, general solution of the equations of motion of a system  $S$  (see Sec. 7.1) because the differential equations governing the generalized coordinates and generalized speeds are both nonlinear and coupled. At times, some of the generalized coordinates and/or generalized speeds can be expressed as explicit, elementary functions of time, whereas the rest cannot be so expressed. Moreover, there exists no general, systematic procedure that facilitates the search for closed-form solutions. To find them, one must take advantage of special features of the problem under consideration.

**Example** In Fig. 7.4.1,  $S$  represents a uniform sphere of radius  $r$  that rolls on the interior surface of a fixed tube  $T$  of radius  $R$ , the axis of  $T$  being vertical. Generalized speeds  $u_1, \dots, u_5$  may be introduced as

$$u_i \triangleq \boldsymbol{\omega} \cdot \mathbf{e}_i \quad (i = 1, 2, 3) \quad (1)$$

$$u_4 \triangleq \mathbf{v} \cdot \mathbf{e}_2 \quad u_5 \triangleq \mathbf{v} \cdot \mathbf{e}_3 \quad (2)$$

where  $\boldsymbol{\omega}$  is the angular velocity of  $S$ ,  $\mathbf{v}$  is the velocity of the center of  $S$ , and  $\mathbf{e}_1, \mathbf{e}_2$ , and  $\mathbf{e}_3$  form a right-handed set of mutually perpendicular unit vectors directed as shown in Fig. 7.4.1. For generalized coordinates one can use, in

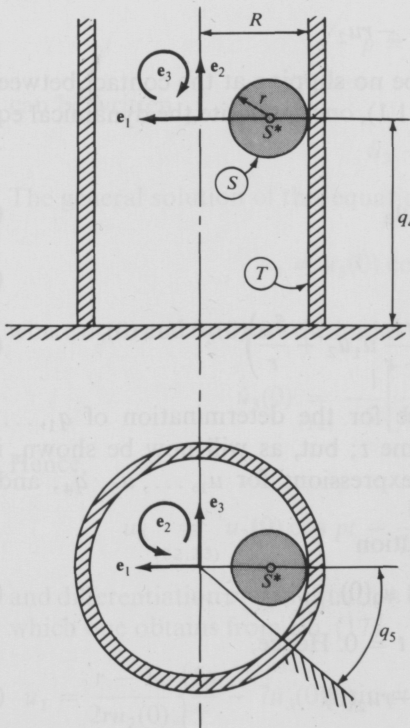


Figure 7.4.1

part, angles  $q_1$ ,  $q_2$ , and  $q_3$ , like those in Problem 1.1, to characterize the orientation of unit vectors  $\mathbf{s}_1$ ,  $\mathbf{s}_2$ ,  $\mathbf{s}_3$  fixed in  $S$  relative to the unit vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $\mathbf{e}_3$ , and, for the rest, the distance  $q_4$  and the angle  $q_5$  shown in Fig. 7.4.1. The kinematical differential equations then are

$$\dot{q}_1 = u_1 + \tan q_2 \left[ \left( u_2 - \frac{u_5}{R-r} \right) s_1 - u_3 c_1 \right] \quad (3)$$

$$\dot{q}_2 = \left( u_2 - \frac{u_5}{R-r} \right) c_1 + u_3 s_1 \quad (4)$$

$$\dot{q}_3 = -\sec q_2 \left[ \left( u_2 - \frac{u_5}{R-r} \right) s_1 - u_3 c_1 \right] \quad (5)$$

$$\dot{q}_4 = u_4 \quad (6)$$

$$\dot{q}_5 = \frac{u_5}{R-r} \quad (7)$$

The system possesses three degrees of freedom because the two nonholonomic constraint equations (see Sec. 2.13)

$$u_4 = ru_3 \quad (8)$$

and

$$u_5 = -ru_2 \quad (9)$$

must be satisfied in order that there be no slipping at the contact between  $S$  and  $T$ . Finally, with the aid of Eqs. (6.1.1), one can write the dynamical equations

$$\dot{u}_1 = \frac{r}{R-r} u_2 u_3 \quad (10)$$

$$\dot{u}_2 = 0 \quad (11)$$

$$\dot{u}_3 = -\frac{1}{7} \left( \frac{2r}{R-r} u_1 u_2 + \frac{5g}{r} \right) \quad (12)$$

In principle, Eqs. (3)–(12) suffice for the determination of  $q_1, \dots, q_5$  and  $u_1, \dots, u_5$  as functions of the time  $t$ ; but, as will now be shown, it is considerably easier to find explicit expressions for  $u_1, \dots, u_5$ ,  $q_4$ , and  $q_5$  than for  $q_1$ ,  $q_2$ , and  $q_3$ .

Equation (11) has the general solution

$$u_2 = u_2(0) \quad (13)$$

where  $u_2(0)$  denotes the value of  $u_2$  at  $t = 0$ . Hence,

$$u_5 = -ru_2(0) \quad (14)$$

(9)

and

$$\dot{q}_5 = \frac{ru_2(0)}{(7,14) r - R} \quad (15)$$

which has the general solution

$$q_5 = \frac{ru_2(0)t}{r - R} + q_5(0) \quad (16)$$

Next, if  $u_2(0) \neq 0$ , then

$$u_1 = \frac{r - R}{(12) 2ru_2(0)} \left( \frac{5g}{r} + 7\dot{u}_3 \right) \quad (17)$$

(13)

so that

$$\dot{u}_1 = \frac{7(r - R)}{(17) 2ru_2(0)} \ddot{u}_3 \quad (18)$$

and

$$\frac{7(r - R)}{2ru_2(0)} \ddot{u}_3 = \frac{r}{(10) R - r} u_2(0) u_3 \quad (19)$$

(18)

which, after  $p$  has been defined as

$$p \triangleq \frac{r|u_2(0)|\sqrt{\frac{2}{7}}}{R - r} \quad (20)$$

can be written

$$\ddot{u}_3 + p^2 u_3 = 0 \quad (21)$$

The general solution of this equation is

$$u_3 = u_3(0) \cos pt + \left[ \frac{\dot{u}_3(0)}{p} \right] \sin pt \quad (22)$$

where  $u_3(0)$  and  $\dot{u}_3(0)$  are the initial values of  $u_3$  and  $\dot{u}_3$ , respectively. Moreover,

$$\dot{u}_3(0) = -\frac{1}{7} \left[ \frac{2r}{R - r} u_1(0) u_2(0) + \frac{5g}{r} \right] \quad (23)$$

(12)

Hence,

$$u_3 = u_3(0) \cos pt - \frac{1}{7p} \left[ \frac{2r}{R - r} u_1(0) u_2(0) + \frac{5g}{r} \right] \sin pt \quad (24)$$

(22,23)

and differentiation of this equation leads to an expression for  $\dot{u}_3$ , with the aid of which one obtains from Eq. (17)

$$u_1 = \frac{r - R}{2ru_2(0)} \left\{ \frac{5g}{r} - 7u_3(0)p \sin pt - \left[ \frac{2r}{R - r} u_1(0) u_2(0) + \frac{5g}{r} \right] \cos pt \right\} \quad (25)$$

As for  $u_4$ , Eqs. (8) and (24) yield

$$u_4 = ru_3(0) \cos pt - \frac{r}{7p} \left[ \frac{2r}{R-r} u_1(0)u_2(0) + \frac{5g}{r} \right] \sin pt \quad (26)$$

Finally, integrating Eq. (6) after replacing  $u_4$  with the right-hand member of Eq. (26), one finds that

$$q_4 = \frac{r}{p} u_3(0) \sin pt + \frac{r}{7p^2} \left[ \frac{2r}{R-r} u_1(0)u_2(0) + \frac{5g}{r} \right] \cos pt + q_4(0) \quad (27)$$

Equations (27), (16), (25), (13), (24), (26), and (14) provide expressions for  $q_4$ ,  $q_5$ ,  $u_1$ ,  $u_2$ ,  $u_3$ ,  $u_4$ , and  $u_5$ , respectively, as explicit functions of time, and thus they furnish much physically meaningful information about the motion of  $S$ . For example, Eq. (16) reveals that  $S^*$  moves in a plane that rotates with a constant angular speed of magnitude  $r|u_2(0)|/(R-r)$  about the axis of  $T$ . The motion of  $S^*$  in this rotating plane is described completely by Eq. (27), which shows that  $S^*$  performs an oscillatory vertical motion whose circular frequency,  $p$  [see Eq. (20)], depends both on the initial value of  $u_2$  and on the geometric parameter  $R/r$ , while the amplitude of the oscillations can be adjusted to any desired value by a suitable choice of the initial values of  $u_1$ ,  $u_2$ , and  $u_3$ . In particular, one can make the amplitude equal to zero—that is, one can make  $S^*$  move on a horizontal circle—by taking

$$u_3(0) = 0 \quad u_1(0) = -\frac{5g(R-r)}{2u_2(0)r^2} \quad (28)$$

These results may conflict with one's intuition; one might expect the sphere to move downward, regardless of initial conditions. Indeed, this is what happens in reality. The reason for this discrepancy between predicted and actual motions is that certain physically unavoidable, dissipative effects, such as frictional resistance to rotation, have been left out of account in the analysis. Problem 13.9 deals with a more realistic approach.

When the expressions for  $u_1$ ,  $u_2$ , and  $u_3$  available in Eqs. (25), (13), and (24), respectively, are substituted into Eqs. (3)–(5), it can be seen that  $q_1$ ,  $q_2$ , and  $q_3$  are governed by a set of coupled, nonlinear differential equations with time-dependent coefficients. Generally, solutions of such sets of equations can be found only by numerical integration.

## 7.5 NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS OF MOTION

When the dynamical (see Sec. 6.1) and/or kinematical (see Sec. 2.12) differential equations governing the motion of a system  $S$  cannot be solved in closed form (see Sec. 7.4), one can use a numerical integration procedure to determine for time  $t \geq t_0$  the values of the dependent variables appearing in the equations, provided that the values of these variables are known for  $t = t_0$ ,  $t_0$  being a particular value

of  $t$ . To this end, one can employ any one of many computer programs applicable to the solution of a set of  $v$  first-order differential equations of the form

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_v; t) \quad (i = 1, \dots, v) \quad (1)$$

where  $x_1, \dots, x_v$  are unknown functions of  $t$ , and  $f_1, \dots, f_v$  are known, generally nonlinear functions of  $x_1, \dots, x_v$ , and  $t$ .

Sometimes the differential equations of motion of a system are available precisely in the form of Eqs. (1). When this is the case, one can proceed directly to the numerical solution of the equations, taking care in programming to ensure that the evaluation of any expression occurring repeatedly in  $f_1, \dots, f_v$  is performed only once. More frequently, at least the dynamical differential equations (see Sec. 6.1) are not immediately available in the form of Eqs. (1), for use of Eqs. (6.1.1) or (6.1.2) leads to equations of the form

$$X\dot{u} = Y \quad (2)$$

where  $\dot{u}$  is a  $p \times 1$  matrix [ $p = n$  when Eqs. (6.1.2) apply] having the time-derivative  $\dot{u}_r$  of the generalized speed  $u_r$  as the  $r$ th element ( $r = 1, \dots, p$ ),  $X$  is a  $p \times p$  matrix whose elements are functions of the generalized coordinates  $q_1, \dots, q_n$  and the time  $t$ , and  $Y$  is a  $p \times 1$  matrix whose elements are functions of  $q_1, \dots, q_n, u_1, \dots, u_n$ , and  $t$ . Under these circumstances, one must “uncouple” Eqs. (2), that is, one must solve these equations for  $\dot{u}_1, \dots, \dot{u}_p$ , in order to be able to use a computer program intended for the solution of Eqs. (1). When  $p$  is sufficiently small, or when  $X$  is sufficiently sparse, one can do this in analytical terms; otherwise, one must resort to the use of a computer routine that solves a set of linear equations numerically, calling this routine each time  $\dot{u}_1, \dots, \dot{u}_p$  are needed.

When one undertakes the task of forming  $X$  and  $Y$ , that is, of formulating dynamical differential equations, with a view toward performing a numerical simulation of a motion of a system  $S$ , one can take certain measures that save one considerable amounts of writing and that lead to highly efficient computer codes for the evaluation of  $X$  and  $Y$ , particularly when  $p$  is large. The two guiding ideas here are that  $\dot{u}_1, \dots, \dot{u}_p$  must be kept in explicit evidence, and that repeated writing of any expression must be avoided. The introduction of auxiliary constants  $k_1, k_2, \dots$ , and auxiliary functions  $Z_1, Z_2, \dots$  of  $q_1, \dots, q_n, u_1, \dots, u_n$ , and  $t$ , enables one to reach both of these goals, as will be illustrated in connection with an example.

**Example** Figure 7.5.1 is a schematic representation of a spacecraft equipped with a nutation damper. This system consists of a rigid body  $B$  and a particle  $P$  that moves in a tube  $T$ , with  $P$  attached to  $B$  by means of a light, linear spring  $S$  and a light, linear dashpot  $D$ . The axis of  $T$  is parallel to line  $L_1$ , one of the central principal axes of  $B$ ; this axis is separated from  $L_1$  by a distance  $b$ ; and it lies in the  $L_1L_2$  plane, where  $L_2$ , like  $L_1$ , is a central principal axis of  $B$ . The force  $\mathbf{R}$  exerted on  $P$  by  $S$  and  $D$  is given by

$$\mathbf{R} = -(\sigma q + \delta \dot{q})\mathbf{b}_1 \quad (3)$$

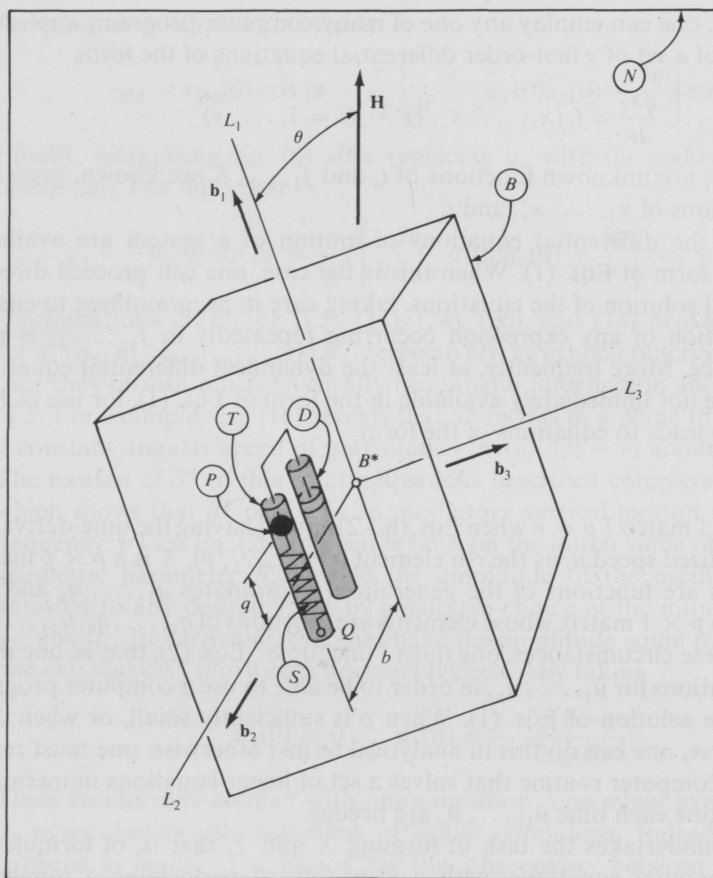


Figure 7.5.1

where  $\sigma$  and  $\delta$  are constants,  $q$  is the distance from  $L_2$  to  $P$ , and  $\mathbf{b}_1$  is a unit vector parallel to  $L_1$ .

When the spacecraft moves in a Newtonian reference frame  $N$  in the absence of external forces, then  $\mathbf{H}$ , the central angular momentum of the spacecraft, remains fixed in  $N$ , but  $\theta$ , the angle between  $L_1$  and  $\mathbf{H}$ , varies with time. The time-history of  $\theta$  is to be determined.

Once  $\mathbf{H}$  is known as a function of  $t$ ,  $\theta$  can be found with the aid of the relationship

$$\theta = \cos^{-1} \left( \mathbf{b}_1 \cdot \frac{\mathbf{H}}{|\mathbf{H}|} \right) \quad (4)$$

As for  $\mathbf{H}$ , Eqs. (3.5.27), (3.5.28), and the theorem stated in Problem 6.3 lead to

$$\mathbf{H} = \mathbf{I} \cdot {}^N \boldsymbol{\omega}^B + m_B \mathbf{p}^{CB*} \times {}^N \mathbf{v}^{B*} + m_P \mathbf{p}^{CP} \times {}^N \mathbf{v}^P \quad (5)$$

where  $\mathbf{I}$  is the central inertia dyadic of  $B$ ,  ${}^N\boldsymbol{\omega}^B$  is the angular velocity of  $B$  in  $N$ ,  $m_B$  and  $m_P$  are the masses of  $B$  and  $P$ , respectively,  $C$  denotes the mass center of the system formed by  $B$  and  $P$ ,  $\mathbf{p}^{CB^*}$  and  $\mathbf{p}^{CP}$  are, respectively, the position vectors from  $C$  to  $B^*$ , the mass center of  $B$ , and  $C$  to  $P$ , and  ${}^N\mathbf{v}^{B^*}$  and  ${}^N\mathbf{v}^P$  are, respectively, the velocities of  $B^*$  and  $P$  in  $N$ . Hence, we must determine  ${}^N\boldsymbol{\omega}^B$ ,  $\mathbf{p}^{CB^*}$ ,  ${}^N\mathbf{v}^{B^*}$ ,  $\mathbf{p}^{CP}$ , and  ${}^N\mathbf{v}^P$  as functions of  $t$ . To this end, we shall introduce generalized speeds  $u_1, \dots, u_7$  in such a way that each of the required vectors can be expressed as a function of  $u_1, \dots, u_7$ , and  $q$ ; formulate generalized inertia forces  $F_1^*, \dots, F_7^*$  and generalized active forces  $F_1, \dots, F_7$ ; use Eqs. (6.1.2) to write dynamical differential equations governing  $u_1, \dots, u_7$ ; and solve these numerically together with a kinematical differential equation that will be stated presently, thus obtaining  $u_1, \dots, u_7$ , and  $q$  as functions of  $t$ .

The generalized speeds to be employed are defined as

$$u_i \triangleq {}^N\boldsymbol{\omega}^B \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \quad (6)$$

$$u_4 \triangleq \dot{q} \quad (7)$$

$$u_{4+i} \triangleq {}^N\mathbf{v}^{B^*} \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \quad (8)$$

where  $\mathbf{b}_2$  and  $\mathbf{b}_3$  are unit vectors, with  $\mathbf{b}_2$  parallel to  $L_2$ , and  $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ . Equation (7) is the aforementioned kinematical differential equation, and the vectors appearing in Eq. (5) can be expressed in terms of  $u_1, \dots, u_7$ , and  $q$  as follows:

$${}^N\boldsymbol{\omega}^B = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_3 \mathbf{b}_3 \quad (9)$$

$$\mathbf{p}^{CB^*} = - \frac{m_P}{m_B + m_P} (q \mathbf{b}_1 + b \mathbf{b}_2) \quad (3.1.2) \quad (10)$$

$${}^N\mathbf{v}^{B^*} = u_5 \mathbf{b}_1 + u_6 \mathbf{b}_2 + u_7 \mathbf{b}_3 \quad (8) \quad (11)$$

$$\mathbf{p}^{CP} = \frac{m_B}{m_B + m_P} (q \mathbf{b}_1 + b \mathbf{b}_2) \quad (3.1.2) \quad (12)$$

and [use Eqs. (2.8.1), (2.7.1), (7), and (9)–(12)]

$${}^N\mathbf{v}^P = (u_5 + u_4 - u_3 b) \mathbf{b}_1 + (u_6 + u_3 q) \mathbf{b}_2 + (u_7 + u_1 b - u_2 q) \mathbf{b}_3 \quad (13)$$

Reference to Eqs. (4.11.2), (4.11.3), (4.11.6), and (4.11.7) allows one to express  $F_r^*$  as

$$F_r^* = {}^N\boldsymbol{\omega}^{B^*} \cdot \mathbf{T}^* - m_B {}^N\mathbf{v}_r^{B^*} \cdot {}^N\mathbf{a}^{B^*} - m_P {}^N\mathbf{v}_r^P \cdot {}^N\mathbf{a}^P \quad (r = 1, \dots, 7) \quad (14)$$

where  $\mathbf{T}^*$  is the inertia torque for  $B$  in  $N$ ,  ${}^N\mathbf{a}^{B^*}$  and  ${}^N\mathbf{a}^P$  are the accelerations of  $B^*$  and  $P$  in  $N$ , and  ${}^N\boldsymbol{\omega}_r^B$ ,  ${}^N\mathbf{v}_r^{B^*}$ , and  ${}^N\mathbf{v}_r^P$  ( $r = 1, \dots, 7$ ) are partial angular velocities of  $B$  in  $N$  and partial velocities of  $B^*$  and  $P$  in  $N$ . As for  $F_r$ , since it is presumed that the spacecraft moves in  $N$  in the absence of external forces, the only forces that contribute to  $F_r$  are the force  $\mathbf{R}$  given by Eq. (3) and the

reaction to this force, that is, a force  $-\mathbf{R}$  applied to  $B$  at some point along the axis of  $T$ , such as the point  $Q$  shown in Fig. 7.5.1. Hence,  $F_r$  is given by

$$F_r = {}^N\mathbf{v}_r^P \cdot \mathbf{R} + {}^N\mathbf{v}_r^Q \cdot (-\mathbf{R}) \quad (r = 1, \dots, 7) \quad (15)$$

where  ${}^N\mathbf{v}_r^Q$  ( $r = 1, \dots, 7$ ) are partial velocities of  $Q$  in  $N$ .

Equations (14) and (15) show that the partial angular velocities of  $B$  and the partial velocities of  $B^*$ ,  $P$ , and  $Q$  are needed. Placing  $Q$  (see Fig. 7.5.1) at a distance  $d$  from  $L_2$  (see Fig. 7.5.1), one can write

$${}^N\mathbf{v}^Q = {}^N\mathbf{v}^{B^*} + {}^N\boldsymbol{\omega}^B \times (-d\mathbf{b}_1 + b\mathbf{b}_2) \quad (16)$$

or, in view of Eqs. (9) and (11),

$${}^N\mathbf{v}^Q = (u_5 - u_3b)\mathbf{b}_1 + (u_6 - u_3d)\mathbf{b}_2 + (u_7 + u_1b + u_2d)\mathbf{b}_3 \quad (17)$$

and this brings one into position to record all necessary partial angular velocities and partial velocities as in Table 7.5.1, where numbers in parentheses in column headings refer to equations numbered correspondingly.

The next item required for substitution into Eqs. (14) is the inertia torque  $\mathbf{T}^*$ , which can be written

$$\begin{aligned} \mathbf{T}^* &= \underset{(4.11.10,6)}{-} [\dot{u}_1 B_1 - u_2 u_3 (B_2 - B_3)] \mathbf{b}_1 \\ &\quad - [\dot{u}_2 B_2 - u_3 u_1 (B_3 - B_1)] \mathbf{b}_2 \\ &\quad - [\dot{u}_3 B_3 - u_1 u_2 (B_1 - B_2)] \mathbf{b}_3 \end{aligned} \quad (18)$$

where  $B_1$ ,  $B_2$ , and  $B_3$  are the moments of inertia of  $B$  about lines  $L_1$ ,  $L_2$ , and  $L_3$  (see Fig. 7.5.1), respectively. At this point, the process of simplifying the analysis that follows can be begun through the introduction of constants  $k_1$ ,  $k_2$ ,  $k_3$  as

$$k_1 \triangleq B_2 - B_3 \quad k_2 \triangleq B_3 - B_1 \quad k_3 \triangleq B_1 - B_2 \quad (19)$$

and three functions of  $u_1$ ,  $u_2$ , and  $u_3$  as

$$Z_1 \triangleq u_2 u_3 k_1 \quad Z_2 \triangleq u_3 u_1 k_2 \quad Z_3 \triangleq u_1 u_2 k_3 \quad (20)$$

**Table 7.5.1**

$r$	${}^N\boldsymbol{\omega}_r^B$ (9)	${}^N\mathbf{v}_r^{B^*}$ (11)	${}^N\mathbf{v}_r^P$ (13)	${}^N\mathbf{v}_r^Q$ (17)
1	$\mathbf{b}_1$	0	$b\mathbf{b}_3$	$b\mathbf{b}_3$
2	$\mathbf{b}_2$	0	$-q\mathbf{b}_3$	$d\mathbf{b}_3$
3	$\mathbf{b}_3$	0	$-b\mathbf{b}_1 + q\mathbf{b}_2$	$-b\mathbf{b}_1 - d\mathbf{b}_2$
4	0	0	$\mathbf{b}_1$	0
5	0	$\mathbf{b}_1$	$\mathbf{b}_1$	$\mathbf{b}_1$
6	0	$\mathbf{b}_2$	$\mathbf{b}_2$	$\mathbf{b}_2$
7	0	$\mathbf{b}_3$	$\mathbf{b}_3$	$\mathbf{b}_3$



which makes it possible to replace Eq. (18) with

$$\mathbf{T}^* = (-\dot{u}_1 B_1 + Z_1)\mathbf{b}_1 + (-\dot{u}_2 B_2 + Z_2)\mathbf{b}_2 + (-\dot{u}_3 B_3 + Z_3)\mathbf{b}_3 \quad (21)$$

(18-20)

The reason for introducing  $k_1, k_2, k_3$  as in Eqs. (19) and then expressing  $Z_1, Z_2, Z_3$  in terms of  $k_1, k_2, k_3$ , rather than simply defining  $Z_1$  as  $u_2 u_3 (B_2 - B_3)$ , and similarly for  $Z_2$  and  $Z_3$ , is that  $k_1, k_2$ , and  $k_3$ , being constants, can be evaluated once and for all, whereas  $Z_1, Z_2$ , and  $Z_3$ , being functions of the dependent variables  $u_1, u_2$ , and  $u_3$ , must be evaluated repeatedly in the course of a numerical integration. Hence, if  $Z_1$  were defined as  $u_2 u_3 (B_2 - B_3)$ , the subtraction of  $B_3$  from  $B_2$  would have to be performed repeatedly, but if  $Z_1$  is defined as in the first of Eqs. (20), then this subtraction takes place only once.

To form suitable expressions for  ${}^N \mathbf{a}^{B^*}$  and  ${}^N \mathbf{a}^P$ , the two vectors in Eqs. (14) that have not yet been considered in detail, we differentiate Eqs. (11) and (13) with respect to  $t$  in  $N$ , obtaining [see Eqs. (2.6.2) and (2.3.1)] in the first case

$${}^N \mathbf{a}^{B^*} = \dot{u}_5 \mathbf{b}_1 + \dot{u}_6 \mathbf{b}_2 + \dot{u}_7 \mathbf{b}_3 + {}^N \boldsymbol{\omega}^B \times {}^N \mathbf{v}^{B^*} \quad (22)$$

(11)

or, after using Eq. (9),

$$\begin{aligned} {}^N \mathbf{a}^{B^*} = & (\dot{u}_5 + u_2 u_7 - u_3 u_6) \mathbf{b}_1 + (\dot{u}_6 + u_3 u_5 - u_1 u_7) \mathbf{b}_2 \\ & + (\dot{u}_7 + u_1 u_6 - u_2 u_5) \mathbf{b}_3 \end{aligned} \quad (23)$$

(9,11)

which we simplify by letting

$$Z_4 \triangleq u_2 u_7 - u_3 u_6 \quad (24)$$

$$Z_5 \triangleq u_3 u_5 - u_1 u_7 \quad (25)$$

$$Z_6 \triangleq u_1 u_6 - u_2 u_5 \quad (26)$$

whereupon we have

$${}^N \mathbf{a}^{B^*} = (\dot{u}_5 + Z_4)\mathbf{b}_1 + (\dot{u}_6 + Z_5)\mathbf{b}_2 + (\dot{u}_7 + Z_6)\mathbf{b}_3 \quad (27)$$

(23)                      (24)                      (25)                      (26)

Similarly,

$$\begin{aligned} {}^N \mathbf{a}^P = & (\dot{u}_5 + \dot{u}_4 - \dot{u}_3 b) \mathbf{b}_1 + (\dot{u}_6 + \dot{u}_3 q + u_3 \dot{q}) \mathbf{b}_2 \\ & + (\dot{u}_7 + \dot{u}_1 b - \dot{u}_2 q - u_2 \dot{q}) \mathbf{b}_3 + {}^N \boldsymbol{\omega}^B \times {}^N \mathbf{v}^P \end{aligned} \quad (28)$$

(13)

and here it is advantageous to introduce  $Z$ 's that not only permit one to write the last term in a simple form, but that at the same time accommodate those portions of the  $\mathbf{b}_2$  and  $\mathbf{b}_3$  measure numbers of  ${}^N \mathbf{a}^P$  that do not involve  $\dot{u}_1, \dots, \dot{u}_7$ , that is, the two terms  $u_3 \dot{q}$  and  $-u_2 \dot{q}$ , which, incidentally, are

respectively equal to  $u_3 u_4$  and  $-u_2 u_4$ , as can be seen by reference to Eq. (7). Specifically, we let [see Eqs. (13) and (9)]

$$Z_7 \triangleq u_5 + u_4 - u_3 b \quad (29)$$

$$Z_8 \triangleq u_6 + u_3 q \quad (30)$$

$$Z_9 \triangleq u_7 + u_1 b - u_2 q \quad (31)$$

$$Z_{10} \triangleq Z_7 + u_4 \quad (32)$$

$$Z_{11} \triangleq u_2 Z_9 - u_3 Z_8 \quad (33)$$

$$Z_{12} \triangleq u_3 Z_7 - u_1 Z_9 + u_3 u_4 \underset{(32)}{=} u_3 Z_{10} - u_1 Z_9 \quad (34)$$

$$Z_{13} \triangleq u_1 Z_8 - u_2 Z_7 - u_2 u_4 \underset{(32)}{=} u_1 Z_8 - u_2 Z_{10} \quad (35)$$

and this enables us to express  ${}^N \mathbf{a}^P$  as

$$\begin{aligned} {}^N \mathbf{a}^P &= \underset{(28-35)}{(\dot{u}_5 + \dot{u}_4 - \dot{u}_3 b + Z_{11})} \mathbf{b}_1 \\ &\quad + (\dot{u}_6 + \dot{u}_3 q + Z_{12}) \mathbf{b}_2 \\ &\quad + (\dot{u}_7 + \dot{u}_1 b - \dot{u}_2 q + Z_{13}) \mathbf{b}_3 \end{aligned} \quad (36)$$

Equations (14) and Table 7.5.1 now permit us to write

$$F_1^* = -\dot{u}_1 B_1 + Z_1 - m_P b (\dot{u}_7 + \dot{u}_1 b - \dot{u}_2 q + Z_{13}) \underset{(21)}{\quad} \underset{(36)}{\quad} \quad (37)$$

$$F_2^* = -\dot{u}_2 B_2 + Z_2 + m_P q (\dot{u}_7 + \dot{u}_1 b - \dot{u}_2 q + Z_{13}) \underset{(21)}{\quad} \underset{(36)}{\quad} \quad (38)$$

$$\begin{aligned} F_3^* &= -\dot{u}_3 B_3 + Z_3 + m_P b (\dot{u}_5 + \dot{u}_4 - \dot{u}_3 b + Z_{11}) \underset{(21)}{\quad} \underset{(36)}{\quad} \\ &\quad - m_P q (\dot{u}_6 + \dot{u}_3 q + Z_{12}) \end{aligned} \quad (39)$$

$$F_4^* = -m_P (\dot{u}_5 + \dot{u}_4 - \dot{u}_3 b + Z_{11}) \underset{(36)}{\quad} \quad (40)$$

$$F_5^* = -m_B (\dot{u}_5 + Z_4) - m_P (\dot{u}_5 + \dot{u}_4 - \dot{u}_3 b + Z_{11}) \underset{(27)}{\quad} \underset{(36)}{\quad} \quad (41)$$

$$F_6^* = -m_B (\dot{u}_6 + Z_5) - m_P (\dot{u}_6 + \dot{u}_3 q + Z_{12}) \underset{(27)}{\quad} \underset{(36)}{\quad} \quad (42)$$

$$F_7^* = -m_B (\dot{u}_7 + Z_6) - m_P (\dot{u}_7 + \dot{u}_1 b - \dot{u}_2 q + Z_{13}) \underset{(27)}{\quad} \underset{(36)}{\quad} \quad (43)$$

and detailed expressions for  $F_1, \dots, F_7$  can be formed by substituting from Table 7.5.1 and Eq. (3) into Eqs. (15), with  $\dot{q}$  replaced by  $u_4$  [see Eq. (7)] in Eq. (3). This leads to

$$F_1 = F_2 = F_3 = F_5 = F_6 = F_7 = 0 \quad (44)$$

$$F_4 = -(\sigma q + \delta u_4) \quad (45)$$

Before using Eqs. (37)–(45) in conjunction with Eqs. (6.1.2) to write dynamical differential equations, we note that certain combinations of constants and functions appear repeatedly in Eqs. (37)–(43). Accordingly, we define  $k_4, \dots, k_8$  and  $Z_{14}, Z_{15}, Z_{16}$  as

$$k_4 \triangleq m_P b \quad (46)$$

$$k_5 \triangleq k_4 b \quad (47)$$

$$k_6 \triangleq B_1 + k_5 \quad (48)$$

$$k_7 \triangleq B_3 + k_5 \quad (49)$$

$$k_8 \triangleq m_B + m_P \quad (50)$$

and

$$Z_{14} \triangleq m_P q \quad (51)$$

$$Z_{15} \triangleq Z_{14} q \quad (52)$$

$$Z_{16} \triangleq k_4 q \quad (53)$$

whereupon we find that Eqs. (6.1.2) and (37)–(53) yield

$$-k_6 \dot{u}_1 + Z_{16} \dot{u}_2 - k_4 \dot{u}_7 = -Z_1 + k_4 Z_{13} \quad (54)$$

$$Z_{16} \dot{u}_1 - (B_2 + Z_{15}) \dot{u}_2 + Z_{14} \dot{u}_7 = -Z_2 - Z_{14} Z_{13} \quad (55)$$

$$-(k_7 + Z_{15}) \dot{u}_3 + k_4 \dot{u}_4 + k_4 \dot{u}_5 - Z_{14} \dot{u}_6 = -Z_3 - k_4 Z_{11} + Z_{14} Z_{12} \quad (56)$$

$$k_4 \dot{u}_3 - m_P \dot{u}_4 - m_P \dot{u}_5 = m_P Z_{11} + \sigma q + \delta u_4 \quad (57)$$

$$k_4 \dot{u}_3 - m_P \dot{u}_4 - k_8 \dot{u}_5 = m_B Z_4 + m_P Z_{11} \quad (58)$$

$$-Z_{14} \dot{u}_3 - k_8 \dot{u}_6 = m_B Z_5 + m_P Z_{12} \quad (59)$$

$$-k_4 \dot{u}_1 + Z_{14} \dot{u}_2 - k_8 \dot{u}_7 = m_B Z_6 + m_P Z_{13} \quad (60)$$

and these equations have precisely the form of Eqs. (2) if the elements  $X_{ij}$  and  $Y_i$  ( $i, j = 1, \dots, 7$ ) of  $X$  and  $Y$  are taken to be (only nonzero elements are displayed)

$$\begin{array}{lll} X_{11} = -k_6 & X_{12} = X_{21} = Z_{16} & X_{17} = X_{71} = -k_4 \end{array} \quad \begin{array}{l} (54) \\ (54,55) \\ (54,60) \end{array} \quad (61)$$

$$\begin{array}{lll} X_{22} = -(B_2 + Z_{15}) & X_{27} = X_{72} = Z_{14} & \end{array} \quad \begin{array}{l} (55) \\ (55,60) \end{array} \quad (62)$$

$$\begin{array}{lll} X_{33} = -(k_7 + Z_{15}) & X_{34} = X_{43} = k_4 & \end{array} \quad \begin{array}{l} (56) \\ (56,57) \end{array}$$

$$\begin{array}{lll} X_{35} = X_{53} = k_4 & X_{36} = X_{63} = -Z_{14} & \end{array} \quad \begin{array}{l} (56,58) \\ (56,59) \end{array} \quad (63)$$

$$\begin{array}{lll} X_{44} = -m_P & X_{45} = X_{54} = -m_P & \end{array} \quad \begin{array}{l} (57) \\ (57,58) \end{array} \quad (64)$$

$$\begin{array}{lll} X_{55} = -k_8 & X_{66} = -k_8 & X_{77} = -k_8 \end{array} \quad \begin{array}{l} (58) \\ (59) \\ (60) \end{array} \quad (65)$$

and

$$Y_1 = -Z_1 + k_4 Z_{13} \quad (66)$$

$$Y_2 = -Z_2 - Z_{14} Z_{13} \quad (67)$$

$$Y_3 = -Z_3 - k_4 Z_{11} + Z_{14} Z_{12} \quad (68)$$

$$Y_4 = m_P Z_{11} + \sigma q + \delta u_4 \quad (69)$$

$$Y_5 = m_B Z_4 + m_P Z_{11} \quad (70)$$

$$Y_6 = m_B Z_5 + m_P Z_{12} \quad (71)$$

$$Y_7 = m_B Z_6 + m_P Z_{13} \quad (72)$$

Since the matrix  $X$  is relatively sparse, one can uncouple Eqs. (54)–(60) by hand, proceeding as follows: Solve Eqs. (54), (55), and (60) for  $\dot{u}_1$ ,  $\dot{u}_2$ , and  $\dot{u}_7$ ; solve Eq. (59) for  $\dot{u}_6$ ; substitute into Eq. (56); solve Eqs. (56), (57), and (58) for  $\dot{u}_3$ ,  $\dot{u}_4$ , and  $\dot{u}_5$ . The resulting equations together with Eq. (7) can be written in the form of Eqs. (1), with  $v = 8$ ,  $x_i = u_i$  ( $i = 1, \dots, 7$ ), and  $x_8 = q$ , and a numerical integration of the equations can be performed, once the initial values of  $u_1, \dots, u_7$ , and  $q$ , as well as the values of the parameters  $m_B, B_1, B_2, B_3, b, m_P, \sigma$ , and  $\delta$  have been specified.

Calculations leading to values of  $\theta$  for various values of  $t$  can be performed simultaneously with the numerical integration of the differential equations of motion on the basis of the considerations that follow.

With  $B_1, B_2$ , and  $B_3$  as defined,  $\mathbf{I} \cdot {}^N \boldsymbol{\omega}^B$  is given by

$$\mathbf{I} \cdot {}^N \boldsymbol{\omega}^B = B_1 u_1 \mathbf{b}_1 + B_2 u_2 \mathbf{b}_2 + B_3 u_3 \mathbf{b}_3 \quad (73)$$

and substitution from this equation and Eqs. (10)–(13) into Eq. (5) yields

$$\mathbf{H} = H_1 \mathbf{b}_1 + H_2 \mathbf{b}_2 + H_3 \mathbf{b}_3 \quad (74)$$

where

$$H_1 \triangleq B_1 u_1 + b Z_{17} \quad (75)$$

$$H_2 \triangleq B_2 u_2 - q Z_{17} \quad (76)$$

$$H_3 \triangleq B_3 u_3 + k_9 [u_3 q^2 - b(u_4 - u_3 b)] \quad (77)$$

with

$$k_9 \triangleq \frac{m_B m_P}{k_8} \quad Z_{17} \triangleq k_9 (Z_9 - u_7) \quad (78)$$

Consequently,

$$\theta = \cos^{-1} \frac{H_1}{(H_1^2 + H_2^2 + H_3^2)^{1/2}} \quad (79)$$

The quantity  $(H_1^2 + H_2^2 + H_3^2)^{1/2}$  is of interest not only in this connection, but also because it must remain constant throughout the motion of the spacecraft [set  $\mathbf{M} = 0$  in Eq. (7.3.9) and note Eq. (74)], and its evaluation at various instants during a numerical integration can thus serve as a check on the validity of the integration results.

Turning to some concrete examples in order to illustrate the uses of numerical simulations of motions, we let  $B$  be a uniform rectangular parallelepiped having edges of lengths 1.2 m, 1.225 m, and 1.3 m, these edges being respectively parallel to  $L_1, L_2, L_3$ , and assume that  $B$  has a mass density of 2760 kg/m<sup>3</sup> (the mass density of aluminum), so that  $m_B = 2760 \times 1.2 \times 1.225 \times 1.3 = 5274.4$  kg, while  $B_1 = 5274.4 \times (1.225^2 + 1.3^2)/12 = 1402.4$  kg m<sup>2</sup>,  $B_2 = 5274.4 \times (1.3^2 + 1.2^2)/12 = 1375.7$  kg m<sup>2</sup>,  $B_3 = 5274.4 \times (1.2^2 + 1.225^2)/12 = 1292.5$  kg/m<sup>2</sup>. We take  $b = 1.0$  m and, after defining  $\mu$  as

$$\mu \triangleq \frac{m_P}{m_B} \quad (80)$$

set  $m_P = 52.744$  kg, so that  $\mu = 0.01$ . As for  $\sigma$  and  $\delta$ , we choose these such that the oscillator formed by  $P, S$ , and  $D$  has a circular natural frequency (see Sec. 7.10) of 1 rad/s and is critically damped (see Problem 14.9), that is,  $\sigma = 52.744$  N/m, and  $\delta = 105.49$  Ns/m.

To explore what happens if the spacecraft is subjected to a small disturbance while it is performing a simple spinning motion, we begin by noting that, for all  $t$  prior to the disturbance,

$${}^N\boldsymbol{\omega}^B = \Omega \mathbf{b}_1 \quad (81)$$

where  $\Omega$  is a constant, called the nominal spin rate. The velocity of  $C$ , the mass center of the spacecraft, is given by

$${}^N\mathbf{v}^C = 0 \quad (82)$$

and  $P$  remains on line  $L_2$ , so that

$$q = 0 \quad (83)$$

From Eqs. (9) and (81), it then follows that

$$u_1 = \Omega \quad u_2 = u_3 = 0 \quad (84)$$

and Eqs. (7) and (83) imply that

$$u_4 = 0 \quad (85)$$

Furthermore,

$$\begin{aligned}
 {}^N\mathbf{v}^{B*} & \stackrel{(2.7.1)}{=} {}^N\mathbf{v}^C + {}^N\boldsymbol{\omega}^B \times \mathbf{p}^{CB*} \\
 & \stackrel{(82)}{=} \mathbf{0} + \stackrel{(81)}{\Omega \mathbf{b}_1} \times \left( -\frac{m_P}{m_B + m_P} \stackrel{(10.83)}{b \mathbf{b}_2} \right) \\
 & \stackrel{(80)}{=} -\Omega \frac{b\mu}{1+\mu} \mathbf{b}_3 = -\Omega \frac{b}{101} \mathbf{b}_3
 \end{aligned} \tag{86}$$

so that, considering Eq. (11), we conclude that, for all  $t$  prior to a disturbance,

$$u_5 = u_6 = 0 \tag{87}$$

and

$$u_7 = -\frac{\Omega b}{101} \tag{88}$$

Equations (84), (85), (87), (88), and (83) furnish a precise characterization of the undisturbed motion, and it may be verified with the aid of Eqs. (20), (24)–(26), (29)–(35), and (51)–(53) that the equations of motion, that is, Eqs. (7) and (54)–(60) are, in fact, satisfied when  $u_1, \dots, u_7$ , and  $q$  are given by Eqs. (84), (85), (87), (88), and (83), which means that the postulated motion is physically possible. Equations (75)–(79) show that  $\theta = 0$  during this motion.

To study motions that ensue subsequent to a slight disturbance of the motion under consideration, we perform numerical simulations after assigning the following initial values to  $u_1, \dots, u_7$ , and  $q$ :

$$u_1(0) = \Omega + \varepsilon_1 \tag{89}$$

$$u_i(0) = \varepsilon_i \quad (i = 2, \dots, 6) \tag{90}$$

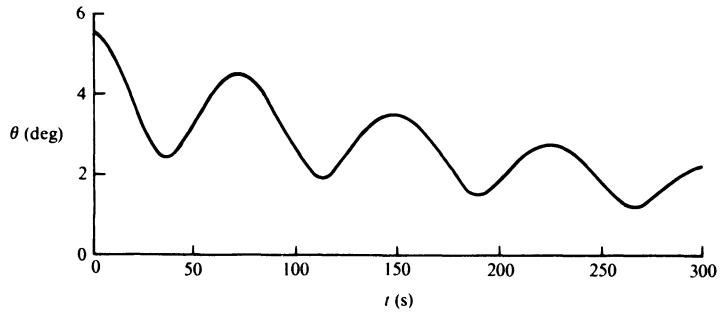
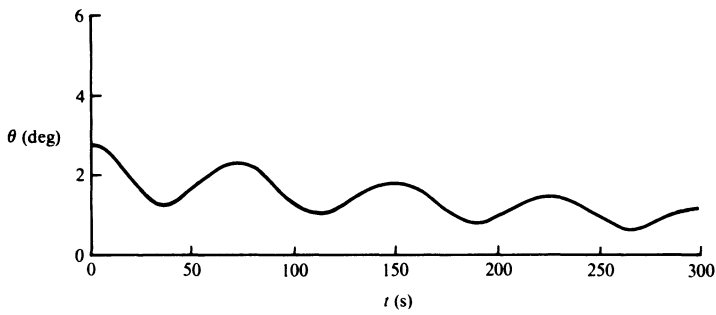
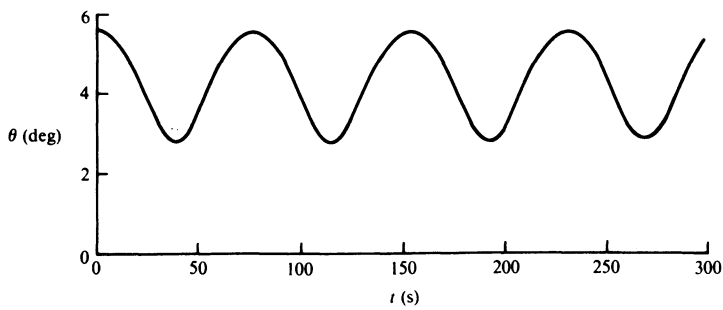
$$u_7(0) = -\frac{\Omega b}{101} + \varepsilon_7 \tag{91}$$

and

$$q(0) = \eta \tag{92}$$

where  $\varepsilon_1, \dots, \varepsilon_7$ , and  $\eta$  are constants called *initial perturbations*. With  $\Omega = 1.0$  rad/s,  $\varepsilon_1 = 0$ ,  $\varepsilon_2 = 0.1$  rad/s,  $\varepsilon_3 = \dots = \varepsilon_7 = \eta = 0$ , this leads to the  $\theta$  versus  $t$  plot displayed in Fig. 7.5.2, which shows that the associated disturbance causes  $\theta$  to have a nonzero initial value and to perform oscillations with decaying amplitudes. When the value of  $\varepsilon_2$  is reduced to 0.05 rad/s, one-half of its former value, one might expect the amplitudes of the oscillations to be smaller than they were previously, and indeed they are, as can be seen in Fig. 7.5.3.

To see that  $P$ ,  $S$ , and  $D$  do, in fact, serve as a nutation *dampener*, one can perform a simulation differing in only one respect from the one that led to Fig. 7.5.2, namely, in that it applies when  $m_P$  is equal to zero. The resulting  $\theta$  versus  $t$  curve, shown in Fig. 7.5.4, reveals that oscillations in  $\theta$  fail to decay when  $m_P = 0$ . In other words, there is no damping under these circumstances.

**Figure 7.5.2****Figure 7.5.3****Figure 7.5.4**

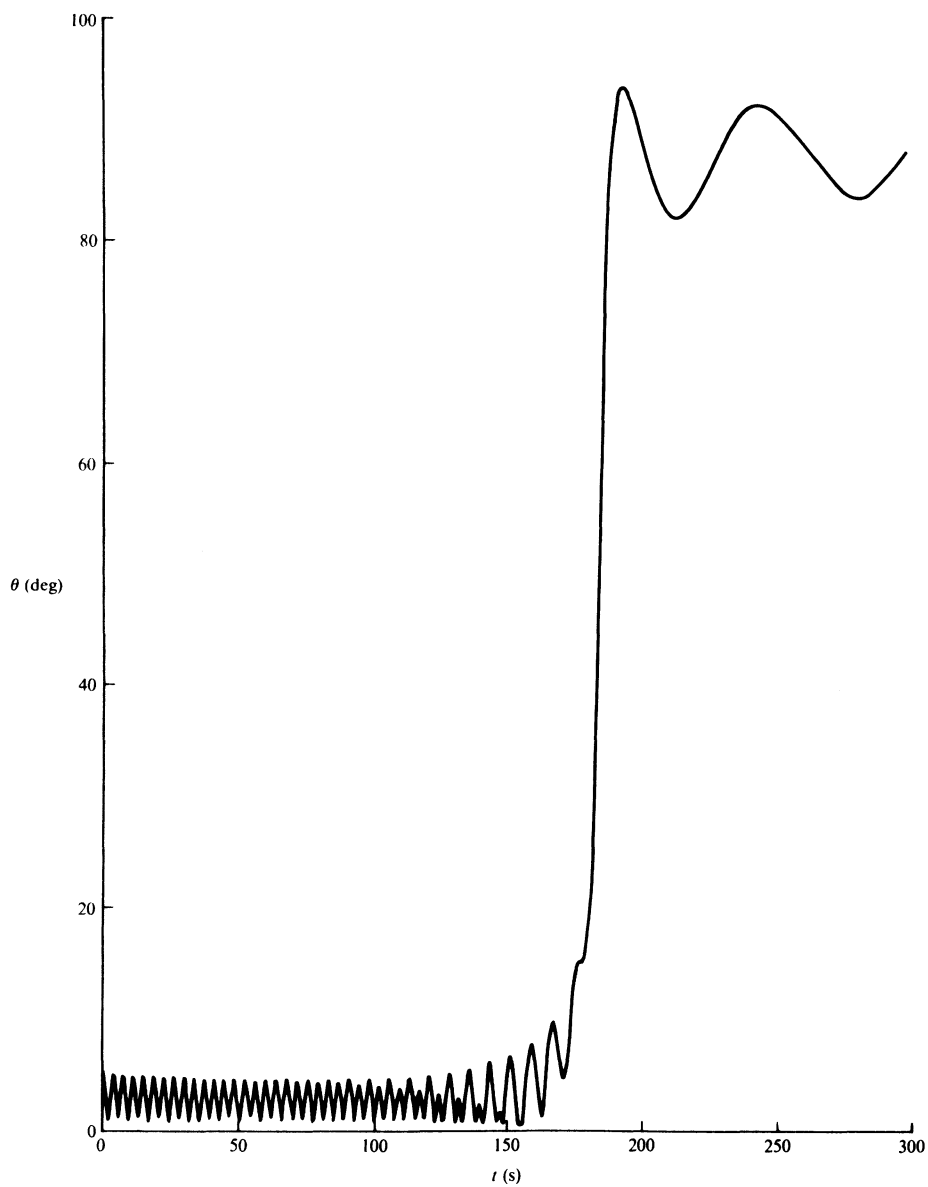


Figure 7.5.5



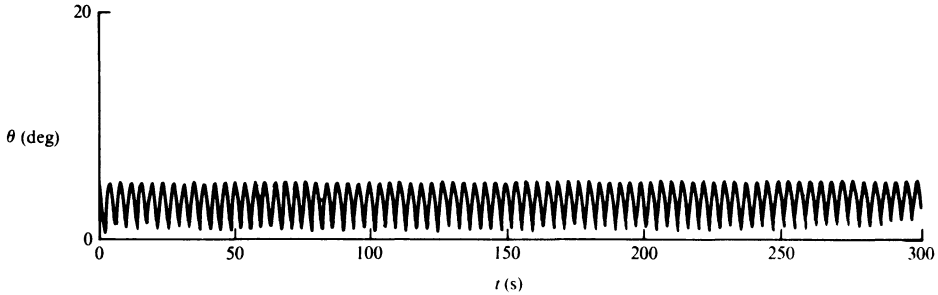


Figure 7.5.6

To see that  $P$ ,  $S$ , and  $D$  can act as a nutation *generator*, rather than as a nutation damper, one can change the dimensions of the edges of  $B$  parallel to  $L_1$ ,  $L_2$ , and  $L_3$  to 0.5 m, 1.2 m, and 3.185 m, respectively, and set  $\sigma = 0.52744$  N/m, so that the volume, and hence the mass, of  $B$  remains unchanged, but  $\sigma$  is reduced to  $\frac{1}{100}$  of its former value. Letting all other quantities have the values used in connection with Fig. 7.5.2, one then obtains Fig. 7.5.5, which portrays highly unstable attitude behavior. Moreover, a simulation differing from the one used to generate Fig. 7.5.5 only in that it applies when  $m_p$  is equal to zero gives rise to Fig. 7.5.6, which leads to the conclusion that  $P$ ,  $S$ , and  $D$  are responsible for the unstable motion depicted in Fig. 7.5.5.

As a check on the validity of the computations underlying Figs. 7.5.2–7.5.6, the quantity  $|\mathbf{H}| \triangleq (H_1^2 + H_2^2 + H_3^2)^{1/2}$  [see Eq. (74)] was evaluated throughout each of the simulations used to produce these figures. In every case,  $|\mathbf{H}|$  was found to remain constant to at least six significant figures.

The preceding results are intended to underscore the fact that simulations of motions can provide valuable physical insights. The extent to which they do so depends heavily on the analyst's ingenuity in choosing parameter values and initial conditions.

## 7.6 DETERMINATION OF CONSTRAINT FORCES AND CONSTRAINT TORQUES

Forces and torques that make no contributions to generalized active forces are called *constraint forces* and *constraint torques*, respectively. To determine these, one uses one or more dynamical equations in which constraint force and/or constraint torque measure numbers of interest come into evidence (see Sec. 6.3), substituting into such equations values of the generalized coordinates and generalized speeds obtained by solving the kinematical and dynamical differential equations governing the motion under consideration.

**Example** Figure 7.6.1 shows a linkage formed by uniform bars  $A$ ,  $B$ , and  $C$ , each of length  $2L$ , having masses  $m_A$ ,  $m_B$ , and  $m_C$ , respectively. Supported

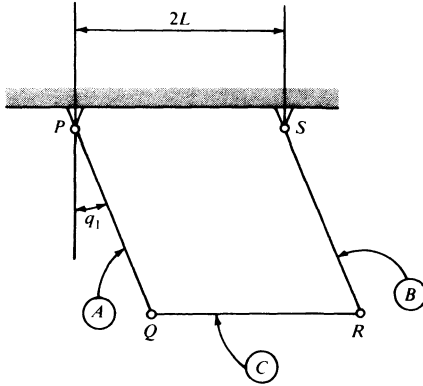


Figure 7.6.1

by horizontal pins at  $P$  and  $S$ , the linkage is free to move in a vertical plane, and such motions are governed by the differential equations

$$\dot{q}_1 = u_1 \quad (1)$$

$$\dot{u}_1 = -p^2 s_1 \quad (2)$$

where  $q_1$  is the radian measure of an angle as indicated in Fig. 7.6.1,  $u_1$  is a generalized speed defined such that Eq. (1) is satisfied,  $s_1 \triangleq \sin q_1$ , and  $p$  is a constant defined as

$$p \triangleq \left[ \frac{3g(m_A + m_B + 2m_C)}{4L(m_A + m_B + 3m_C)} \right]^{1/2} \quad (3)$$

The set of forces exerted on  $A$  by the pin at  $P$  can be replaced with a couple whose torque is perpendicular to  $\mathbf{n}_3$  (see Fig. 7.6.2 for  $\mathbf{n}_3$ ), together with a force  $R_1\mathbf{n}_1 + R_2\mathbf{n}_2 + R_3\mathbf{n}_3$  applied to  $A$  at  $P$ .  $R_1$  is to be determined for  $t = 9$  s and  $t = 10$  s, with  $m_A = 1$  kg,  $m_B = 2$  kg,  $m_C = 3$  kg,  $L = 1$  m, and  $q_1(0) = 30^\circ$ ,  $u_1(0) = 0$ .

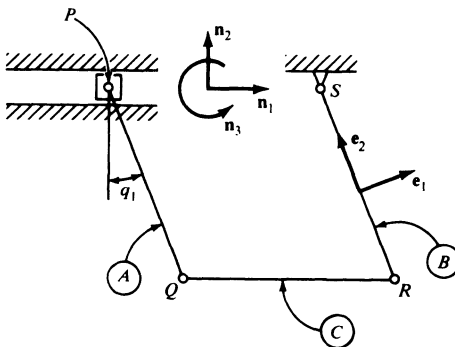


Figure 7.6.2

To bring  $R_1$  (but not  $R_2$ ,  $R_3$ , or the torque of the couple exerted on  $A$  by the pin) into evidence, one can imagine that  $P$  is attached to a slider that is free to move in a horizontal slot, as indicated in Fig. 7.6.2, and one can introduce a generalized speed  $u_2$  such that the velocity of  $P$  is given by (see Fig. 7.6.2 for  $\mathbf{n}_1$ )

$$\mathbf{v}^P = u_2 \mathbf{n}_1 \quad (4)$$

The angular velocities of  $A$ ,  $B$ , and  $C$  then are given by

$$\boldsymbol{\omega}^A = u_1 \mathbf{n}_3 \quad (5)$$

and

$$\boldsymbol{\omega}^B = \beta \mathbf{n}_3 \quad \boldsymbol{\omega}^C = \gamma \mathbf{n}_3 \quad (6)$$

where  $\beta$  and  $\gamma$  are functions of  $q_1$ ,  $u_1$ , and  $u_2$ , which functions are found by setting  $\mathbf{v}^S$ , the velocity of point  $S$ , equal to zero after expressing  $\mathbf{v}^S$  as

$$\mathbf{v}^S \stackrel{(2.7.1)}{=} \mathbf{v}^P + \boldsymbol{\omega}^A \times (-2L\mathbf{e}_2) + \boldsymbol{\omega}^C \times (2L\mathbf{n}_1) + \boldsymbol{\omega}^B \times (2L\mathbf{e}_2) \quad (7)$$

Thus (see Fig. 7.6.2 for the unit vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ ) one obtains

$$0 = \underbrace{u_2}_{(7)} \underbrace{\mathbf{n}_1}_{(4)} + \underbrace{2Lu_1}_{(5)} \underbrace{\mathbf{e}_1}_{(5)} + \underbrace{2L\gamma}_{(6)} \underbrace{\mathbf{n}_2}_{(6)} - \underbrace{2L\beta}_{(6)} \underbrace{\mathbf{e}_1}_{(6)} \quad (8)$$

and dot-multiplications of Eq. (8) with  $\mathbf{n}_1$  and  $\mathbf{e}_2$  lead to

$$\beta = u_1 + \frac{u_2}{2Lc_1} \quad (9)$$

and

$$\gamma = \frac{u_2 s_1}{2Lc_1} \quad (10)$$

respectively, where  $c_1 \triangleq \cos q_1$ . Consequently,

$$\boldsymbol{\omega}^B \stackrel{(6,9)}{=} \left( u_1 + \frac{u_2}{2Lc_1} \right) \mathbf{n}_3 \quad (11)$$

and

$$\boldsymbol{\omega}^C \stackrel{(6,10)}{=} \frac{u_2 s_1}{2Lc_1} \mathbf{n}_3 \quad (12)$$

Furthermore, the velocities of  $A^*$ ,  $B^*$ , and  $C^*$ , the mass centers of  $A$ ,  $B$ , and  $C$ , respectively, are

$$\mathbf{v}^{A^*} = \mathbf{v}^P + \boldsymbol{\omega}^A \times (-L\mathbf{e}_2) = \underset{(4)}{u_2 \mathbf{n}_1} + \underset{(5)}{Lu_1 \mathbf{e}_1} \quad (13)$$

$$\mathbf{v}^{B^*} = \boldsymbol{\omega}^B \times (-L\mathbf{e}_2) = \underset{(11)}{\left(Lu_1 + \frac{u_2}{2c_1}\right)} \mathbf{e}_1 \quad (14)$$

$$\mathbf{v}^{C^*} = \mathbf{v}^Q + \boldsymbol{\omega}^C \times (L\mathbf{n}_1) = \underset{(12)}{2Lu_1 \mathbf{e}_1} + \left(\mathbf{n}_1 + \frac{s_1}{2c_1} \mathbf{n}_2\right) u_2 \quad (15)$$

and with the aid of Eqs. (5), (11), (12), (13)–(15), and (4), one can formulate the partial angular velocities of  $A$ ,  $B$ , and  $C$  and the partial velocities of  $A^*$ ,  $B^*$ ,  $C^*$ , and  $P$  associated with  $u_2$  as

$$\underset{(5)}{\omega_2^A} = 0 \quad \underset{(11)}{\omega_2^B} = \frac{\mathbf{n}_3}{2Lc_1} \quad \underset{(12)}{\omega_2^C} = \frac{s_1 \mathbf{n}_3}{2Lc_1} \quad (16)$$

$$\underset{(13)}{\mathbf{v}_2^{A^*}} = \mathbf{n}_1 \quad \underset{(14)}{\mathbf{v}_2^{B^*}} = \frac{\mathbf{e}_1}{2c_1} \quad \underset{(15)}{\mathbf{v}_2^{C^*}} = \mathbf{n}_1 + \frac{s_1 \mathbf{n}_2}{2c_1} \quad (17)$$

and

$$\underset{(4)}{\mathbf{v}_2^P} = \mathbf{n}_1 \quad (18)$$

With  $u_2 = 0$  (because this corresponds to the motion of interest), the inertia forces for  $A$ ,  $B$ , and  $C$  are [see Eq. (4.11.3)]

$$\mathbf{R}_A^* = -m_A \mathbf{a}^{A^*} = -m_A L(\dot{u}_1 \mathbf{e}_1 + u_1^2 \mathbf{e}_2) \quad (19)$$

$$\mathbf{R}_B^* = -m_B \mathbf{a}^{B^*} = -m_B L(\dot{u}_1 \mathbf{e}_1 + u_1^2 \mathbf{e}_2) \quad (20)$$

and

$$\mathbf{R}_C^* = -m_C \mathbf{a}^{C^*} = -2m_C L(\dot{u}_1 \mathbf{e}_1 + u_1^2 \mathbf{e}_2) \quad (21)$$

respectively, while the inertia torques for  $A$ ,  $B$ , and  $C$  can be written [see Eq. (4.11.10)]

$$\mathbf{T}_A^* = -\frac{m_A L^2}{3} \dot{u}_1 \mathbf{n}_3 \quad (22)$$

$$\mathbf{T}_B^* = -\frac{m_B L^2}{3} \dot{u}_1 \mathbf{n}_3 \quad (23)$$

and

$$\mathbf{T}_C^* = 0 \quad (24)$$

Hence, the generalized inertia force  $F_2^*$ , formed as [see Eq. (4.11.7)]

$$\begin{aligned} F_2^* &= \omega_2^A \cdot \mathbf{T}_A^* + \mathbf{v}_2^{A^*} \cdot \mathbf{R}_A^* + \omega_2^B \cdot \mathbf{T}_B^* \\ &\quad + \mathbf{v}_2^{B^*} \cdot \mathbf{R}_B^* + \omega_2^C \cdot \mathbf{T}_C^* + \mathbf{v}_2^{C^*} \cdot \mathbf{R}_C^* \end{aligned} \quad (25)$$

is given by

$$\begin{aligned}
 F_2^* &= \underset{(25)}{0} \underset{(16,22)}{-m_A L(\dot{u}_1 c_1} \underset{(17,19)}{-u_1^2 s_1)} - \underset{(16,23)}{\frac{m_B L}{6c_1}} \dot{u}_1 - \underset{(17,20)}{\frac{m_B L}{2c_1}} \dot{u}_1 \\
 &+ \underset{(16,24)}{0} - m_C L \left[ \dot{u}_1 \left( c_1 + \frac{1}{c_1} \right) - u_1^2 s_1 \right] \underset{(17,21)}{} \\
 &= L(m_A + m_C) s_1 u_1^2 - L \left\{ m_A c_1 + \frac{1}{c_1} \left[ \frac{2}{3} m_B + (1 + c_1^2) m_C \right] \right\} \dot{u}_1 \quad (26)
 \end{aligned}$$

and the generalized active force  $F_2$ , expressed as [see Eqs. (4.6.1) and (4.8.2)]

$$F_2 = \mathbf{v}_2^P \cdot (R_1 \mathbf{n}_1 + R_2 \mathbf{n}_2 + R_3 \mathbf{n}_3) - g \mathbf{n}_2 \cdot (m_A \mathbf{v}_2^{A^*} + m_B \mathbf{v}_2^{B^*} + m_C \mathbf{v}_2^{C^*}) \quad (27)$$

is given by

$$F_2 = \underset{(27)}{R_1} - g \underset{(17)}{(m_B + m_C)} \frac{s_1}{2c_1} \quad (28)$$

Solved for  $R_1$ , the dynamical equation obtained by substituting from Eqs. (26) and (28) into

$$F_2 + F_2^* \underset{(6.1.2)}{=} 0 \quad (29)$$

yields with the aid of Eq. (2)

$$\begin{aligned}
 R_1 &= g(m_B + m_C) \frac{s_1}{2c_1} - L(m_A + m_C) s_1 u_1^2 \\
 &- L \left\{ m_A c_1 + \frac{1}{c_1} \left[ \frac{2}{3} m_B + (1 + c_1^2) m_C \right] \right\} p^2 s_1 \quad (30)
 \end{aligned}$$

In Table 7.6.1, values are recorded for  $q_1$  and  $u_1$  as found by solving Eqs. (1) and (2) numerically with  $m_A$ ,  $m_B$ ,  $m_C$ ,  $L$ ,  $q_1(0)$ , and  $u_1(0)$  as given. Using the tabulated values corresponding to  $t = 9$  s, one finds that

$$s_1 = -0.1841 \quad c_1 = 0.9829 \quad (31)$$

and

$$\begin{aligned}
 R_1 &= \underset{(30)}{9.81(5)} \frac{(-0.1841)}{2(0.9829)} - (4)(-0.1841)(-1.1358)^2 \\
 &- \left\{ 0.9829 + \frac{1}{0.9829} \left[ \frac{4}{3} + (1 + 0.9661)3 \right] \right\} \underset{(3)}{(5.5181)} (-0.1841) \\
 &= 4.829 \text{ N} \quad (32)
 \end{aligned}$$

**Table 7.6.1**

$t$ (s)	$q_1$ (deg)	$u_1$ (rad/s)
0	30.000	0.0000
1	-20.250	-0.89243
2	-2.8515	1.2103
3	24.051	-0.72140
4	-29.470	-0.22499
5	15.716	1.0325
6	8.4490	-1.1657
7	-26.989	0.52598
8	27.898	0.44270
9	-10.608	-1.1358
10	-13.736	1.0784

Similarly, for  $t = 10$  s, Eq. (30) and the last row of Table 7.6.1 yield  $R_1 = 6.046$  N. Of course, the calculation of  $R_1$  in accordance with Eq. (30) can be incorporated in the computer program used to solve Eqs. (1) and (2).

## 7.7 REAL SOLUTIONS OF A SET OF NONLINEAR, NONDIFFERENTIAL EQUATIONS

The need to find real solutions of a set of nonlinear, nondifferential equations can arise in connection with systems at rest in a Newtonian reference frame, steady motions, motions resembling states of rest, and various other problems of mechanics, as well as other areas of applied mathematics. For instance, referring to the example in Sec. 6.5, one may wish to find a value of the steering angle  $q_4$  such that the system  $S$  remains at rest when  $M, R, a, b, L, T$ , and  $q_5$  have preassigned values. To do so, one must solve Eq. (6.5.31), which one can do easily, despite the fact that the equation is nonlinear in  $q_4$ , for Eq. (6.5.31) leads directly to

$$q_4 = \tan^{-1} \left[ \frac{T - MgR \sin \theta \sin q_5}{Tb/L - (MgRa/L) \sin \theta \cos q_5} \right] \quad (1)$$

Frequently, however, finding such solutions is not so straightforward. Consider, for example, the steady motion problem treated in the example in Sec. 6.6, and suppose that  $\theta$  is to be determined for given values of  $R\Omega^2/g$  and  $h/R$ . Since Eq. (6.6.17) cannot be solved readily for  $\theta$  as a function of  $R\Omega^2/g$  and  $h/R$ , one might

resort to a numerical trial-and-error procedure consisting of evaluating the left-hand member of Eq. (6.6.17) for various values of  $\theta$  until one has found a value that satisfies Eq. (6.6.17) to an acceptable degree of accuracy. But trial-and-error procedures become ineffective when employed for the solution of a set of nonlinear equations, as in the example in Sec. 6.7. Here, to determine  $q_1$  and  $q_2$  when  $L\Omega^2/g$  is given, one must solve Eqs. (6.7.21) and (6.7.22) simultaneously, and one can do this by employing the procedure now to be described.

The most general set of  $n$  simultaneous equations in  $n$  unknowns  $x_1, \dots, x_n$  can be written

$$f_i(x_1, \dots, x_n) = 0 \quad (i = 1, \dots, n) \quad (2)$$

Let  $y_1(\tau), \dots, y_n(\tau)$  be a set of  $n$  functions of a variable  $\tau$ , with  $0 \leq \tau \leq 1$ ; take

$$y_i(0) = k_i \quad (i = 1, \dots, n) \quad (3)$$

where  $k_i$ , a constant, is selected arbitrarily; and require that  $y_1(\tau), \dots, y_n(\tau)$  satisfy the equations

$$f_i(y_1, \dots, y_n) = f_i(k_1, \dots, k_n)(1 - \tau) \quad (i = 1, \dots, n) \quad (4)$$

Then, as the right-hand sides of Eqs. (4) vanish at  $\tau = 1$ , the functions  $y_1(\tau), \dots, y_n(\tau)$  satisfy, at  $\tau = 1$ , precisely the same equations as do  $x_1, \dots, x_n$  [see Eqs. (2)]. Now,  $y_1(1), \dots, y_n(1)$ , and, hence,  $x_1, \dots, x_n$ , may be found as follows: Differentiate Eqs. (4) with respect to  $\tau$ , thus obtaining the set of first-order differential equations

$$\left. \begin{aligned} \frac{\partial f_1}{\partial y_1} \frac{dy_1}{d\tau} + \dots + \frac{\partial f_1}{\partial y_n} \frac{dy_n}{d\tau} &= -f_1(k_1, \dots, k_n) \\ \vdots \\ \frac{\partial f_n}{\partial y_1} \frac{dy_1}{d\tau} + \dots + \frac{\partial f_n}{\partial y_n} \frac{dy_n}{d\tau} &= -f_n(k_1, \dots, k_n) \end{aligned} \right\} \quad (5)$$

and perform a numerical integration of these equations (see Sec. 7.5), using Eqs. (3) as initial conditions and terminating the integration at  $\tau = 1$ .

As was stated previously,  $k_1, \dots, k_n$  may be assigned any values whatsoever. However, it can occur that, for certain choices of  $k_1, \dots, k_n$ , some of  $y_1, \dots, y_n$  do not possess real values for some values of  $\tau$  in the interval  $0 \leq \tau \leq 1$ , in which event the numerical integration of the differential equations cannot be carried to completion. When this happens, one simply changes one or more of  $k_1, \dots, k_n$ . In general, results are obtained most expeditiously when  $k_1, \dots, k_n$  are good approximations to  $x_1, \dots, x_n$ , respectively. Fortunately, in connection with physical problems, one often can make good guesses regarding  $x_1, \dots, x_n$ , and hence assign suitable values to  $k_1, \dots, k_n$ . Finally, it is worth noting that two or more distinct sets of values of  $k_1, \dots, k_n$  can lead to the same values of  $x_1, \dots, x_n$ .

**Example** If  $q_1$ ,  $q_2$ , and  $q_3/L$  in Problem 12.6 are replaced with  $x_1$ ,  $x_2$ , and  $x_3$ , respectively, then  $x_1$ ,  $x_2$ , and  $x_3$  are governed by equations having the form of Eqs. (2) with  $n = 3$  and

$$f_1 \triangleq (1 - s_1)^2 + \left(\frac{3}{2} - s_2\right)^2 + (c_1 - c_2)^2 - \left(x_3 + \frac{1}{4}\right)^2 \quad (6)$$

$$f_2 \triangleq \left(x_3 + \frac{1}{4}\right)s_1 + 2x_3(c_2s_1 - c_1) \quad (7)$$

$$f_3 \triangleq \left(x_3 + \frac{1}{4}\right)s_2 + 2x_3\left(c_1s_2 - \frac{3}{2}c_2\right) \quad (8)$$

where  $s_i$  and  $c_i$  now stand for  $\sin x_i$  and  $\cos x_i$ , respectively ( $i = 1, 2$ ). The equations corresponding to Eqs. (5) are

$$\begin{aligned} 2(s_1c_2 - c_1)\frac{dy_1}{d\tau} + 2\left(c_1s_2 - \frac{3}{2}c_2\right)\frac{dy_2}{d\tau} - 2\left(y_3 + \frac{1}{4}\right)\frac{dy_3}{d\tau} \\ = -\left[(1 - \sin k_1)^2 + \left(\frac{3}{2} - \sin k_2\right)^2\right. \\ \left. + (\cos k_1 - \cos k_2)^2 - \left(k_3 + \frac{1}{4}\right)^2\right] \quad (9) \end{aligned}$$

$$\begin{aligned} \left[\left(y_3 + \frac{1}{4}\right)c_1 + 2y_3(c_1c_2 + s_1)\right]\frac{dy_1}{d\tau} - 2y_3s_1s_2\frac{dy_2}{d\tau} + [s_1 + 2(s_1c_2 - c_1)]\frac{dy_3}{d\tau} \\ = -\left[\left(k_3 + \frac{1}{4}\right)\sin k_1 + 2k_3(\cos k_2 \sin k_1 - \cos k_1)\right] \quad (10) \end{aligned}$$

$$\begin{aligned} -2y_3s_1s_2\frac{dy_1}{d\tau} + \left[\left(y_3 + \frac{1}{4}\right)c_2 + 2y_3\left(c_1c_2 + \frac{3}{2}s_2\right)\right]\frac{dy_2}{d\tau} \\ + \left[s_2 + 2\left(c_1s_2 - \frac{3}{2}c_2\right)\right]\frac{dy_3}{d\tau} \\ = -\left[\left(k_3 + \frac{1}{4}\right)\sin k_2 + 2k_3\left(\cos k_1 \sin k_2 - \frac{3}{2}\cos k_2\right)\right] \quad (11) \end{aligned}$$

where  $s_i$  and  $c_i$  now denote  $\sin y_i$  and  $\cos y_i$ , respectively ( $i = 1, 2$ ).

With  $k_1 = k_2 = k_3 = 0$ , numerical integration of Eqs. (9)–(11) in the interval  $0 \leq \tau \leq 1$  leads to values of  $y_1(1)$ ,  $y_2(1)$ , and  $y_3(1)$ , and thus to values of  $x_1$ ,  $x_2$ , and  $x_3$ , such that

$$q_1 = 35.55^\circ \quad q_2 = 44.91^\circ \quad q_3 = 0.65L \quad (12)$$

Identical results are obtained with  $k_1 = k_2 = 0.1745$  (corresponding to  $q_1 = q_2 = 10^\circ$ ) and  $k_3 = 0.1$ ; but taking  $k_1 = k_2 = -1.745$  and  $k_3 = 2$  produces

$$q_1 = -114.69^\circ \quad q_2 = -94.77^\circ \quad q_3 = 2.91L \quad (13)$$



Equations (12) characterize an equilibrium configuration in which  $B_1$  and  $B_2$  (see Fig. P12.6) lie below the horizontal plane determined by the axes of the pins that support the bars, whereas Eqs. (13) apply when the system is at rest with  $B_1$  and  $B_2$  above this plane.

## 7.8 GENERALIZED IMPULSE, GENERALIZED MOMENTUM

When a system  $S$  is subjected to the action of forces that become very large during a very short time interval, the velocities of certain particles of  $S$  may change substantially during this time interval while the configuration of  $S$  in a Newtonian reference frame  $N$  remains essentially unaltered. This happens, for example, when two relatively inflexible bodies collide with each other. Although such phenomena may appear to be more complex than motions that proceed more smoothly, they frequently can be treated analytically with comparatively simple methods because the presumption that the configuration of  $S$  does not change enables one to integrate Eqs. (6.1.1) or (6.1.2) in general terms and thus to construct a theory involving algebraic rather than differential equations. To this end, two sets of quantities, called generalized impulses and generalized momenta, are defined as follows.

Suppose that  $S$  is a nonholonomic system possessing  $n$  generalized coordinates  $q_1, \dots, q_n$  (see Sec. 2.10) and  $n - m$  independent generalized speeds  $u_1, \dots, u_{n-m}$  (see Sec. 2.12) in  $N$ . Let  $\tilde{\mathbf{v}}_r^{P_i}$  be the  $r$ th nonholonomic partial velocity in  $N$  of a generic particle  $P_i$  of  $S$  (see Sec. 2.14); let  $\mathbf{R}_i$  be the resultant of all contact forces and distance forces acting on  $P_i$ ; and let  $t_1$  and  $t_2$  be the initial and final instants of a time interval such that  $q_1, \dots, q_n$  can be regarded as constant throughout this interval. The *generalized impulse*  $I_r$  is defined as

$$I_r \triangleq \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i}(t_1) \cdot \int_{t_1}^{t_2} \mathbf{R}_i dt \quad (r = 1, \dots, n - m) \quad (1)$$

where  $v$  is the number of particles of  $S$ , and the *generalized momentum*  $p_r$  is defined as

$$p_r(t) \triangleq \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i}(t) \cdot \mathbf{v}^{P_i}(t) \quad (r = 1, \dots, n - m) \quad (2)$$

where  $m_i$  is the mass of  $P_i$  ( $i = 1, \dots, v$ ).

When forming expressions for  $I_r$  ( $r = 1, \dots, n - m$ ) in accordance with Eqs. (1), one regards as negligible the contributions to the time-integral of  $\mathbf{R}_i$  of all forces that remain constant during the time interval beginning at  $t_1$  and ending at  $t_2$ . Moreover,  $I_r$  can be expressed as

$$I_r \approx \int_{t_1}^{t_2} \tilde{F}_r dt \quad (r = 1, \dots, n - m) \quad (3)$$

where  $\tilde{F}_1, \dots, \tilde{F}_{n-m}$  are the nonholonomic generalized active forces for  $S$  in  $N$  (see Sec. 4.4). Consequently, forces that make no contributions to generalized active forces (see Sec. 4.5) also make no contributions to generalized impulses.

The constructing of expressions for generalized momenta often is simplified by the following fact. If  $K$ , the kinetic energy of  $S$  in  $N$  (see Sec. 5.4), is regarded as a function of  $q_1, \dots, q_n, u_1, \dots, u_{n-m}$ , and  $t$  (see Sec. 5.5), then

$$p_r = \frac{\partial K}{\partial u_r} \quad (r = 1, \dots, n - m) \quad (4)$$

The aforementioned integration of Eqs. (6.1.1) from  $t_1$  to  $t_2$  results in the relationships

$$I_r \approx p_r(t_2) - p_r(t_1) \quad (r = 1, \dots, n - m) \quad (5)$$

When  $S$  is a holonomic system,  $m$  is set equal to zero, tildes are omitted from Eqs. (1)–(3), and integration of Eqs. (6.1.2), rather than (6.1.1), leads to Eqs. (5).

**Derivations** In accordance with Eqs. (4.4.1),

$$\int_{t_1}^{t_2} \tilde{F}_r dt = \int_{t_1}^{t_2} \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i dt = \sum_{i=1}^v \int_{t_1}^{t_2} \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i dt \quad (r = 1, \dots, n - m) \quad (6)$$

In general,  $\tilde{\mathbf{v}}_r^{P_i}$  is a function of  $t$  in  $N$ . But if  $t_2 \approx t_1$  and  $q_r(t_2) \approx q_r(t_1)$ , then  $\tilde{\mathbf{v}}_r^{P_i}$  is nearly fixed in  $N$  (and nearly equal to its value at time  $t_1$ ) throughout the time interval beginning at  $t_1$  and ending at  $t_2$ , and

$$\int_{t_1}^{t_2} \tilde{F}_r dt \approx \sum_{(6) i=1}^v \tilde{\mathbf{v}}_r^{P_i}(t_1) \cdot \int_{t_1}^{t_2} \mathbf{R}_i dt = I_r \quad (r = 1, \dots, n - m) \quad (7)$$

in agreement with Eqs. (3).

To establish the validity of Eqs. (4), one can write

$$\begin{aligned} \frac{\partial K}{\partial u_r} &= \frac{\partial}{\partial u_r} \left[ \frac{1}{2} \sum_{i=1}^v m_i (\mathbf{v}^{P_i})^2 \right] = \sum_{i=1}^v m_i \frac{\partial \mathbf{v}^{P_i}}{\partial u_r} \cdot \mathbf{v}^{P_i} \\ &= \sum_{(2.14.4) i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{v}^{P_i} = p_r \quad (r = 1, \dots, n - m) \end{aligned} \quad (8)$$

Finally, integration of Eqs. (4.11.1) produces

$$\begin{aligned} \int_{t_1}^{t_2} \tilde{F}_r^* dt &= \int_{t_1}^{t_2} \sum_{i=1}^v \tilde{\mathbf{v}}_r^{P_i} \cdot (-m_i \mathbf{a}_i) dt \quad (4.11.3) \\ &= - \sum_{i=1}^v m_i \int_{t_1}^{t_2} \tilde{\mathbf{v}}_r^{P_i} \cdot \frac{d\mathbf{v}^{P_i}}{dt} dt \\ &\approx - \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i}(t_1) \cdot \int_{t_1}^{t_2} \frac{d\mathbf{v}^{P_i}}{dt} dt \\ &= - \sum_{i=1}^v m_i \tilde{\mathbf{v}}_r^{P_i}(t_1) \cdot [\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)] \\ &= -[p_r(t_2) - p_r(t_1)] \quad (r = 1, \dots, n - m) \end{aligned} \quad (9)$$

Consequently, integration of Eqs. (6.1.1) yields

$$\int_{t_1}^{t_2} \tilde{F}_r dt + \int_{t_1}^{t_2} \tilde{F}_r^* dt \approx \underset{(7)}{I_r} - \underset{(9)}{[p_r(t_2) - p_r(t_1)]} \underset{(6.1.1)}{=} 0 \quad (r = 1, \dots, n - m) \quad (10)$$

or, equivalently, Eqs. (5).

**Example** Figure 7.8.1 shows a gear train consisting of three identical gears  $G_1$ ,  $G_2$ , and  $G_3$ , each having a radius  $b$  and a moment of inertia  $J$  about its axis. The angular speed  $\Omega$  of  $G_1$  is to be determined on the basis of the assumption that the gears are set into motion suddenly when  $G_1$  is meshed with a gear  $G'$  of radius  $b'$  and axial moment of inertia  $J'$ ,  $G'$  having an angular speed  $\Omega'$  at the instant of first contact between  $G_1$  and  $G'$ .

Before  $G_1$  and  $G'$  are brought into contact, the system  $S$  formed by  $G_1$ ,  $G_2$ ,  $G_3$ , and  $G'$  possesses two degrees of freedom, and, if generalized speeds  $u_1$  and  $u_2$  are defined in terms of  $\omega^{G_1}$  and  $\omega^{G'}$ , the angular velocities of  $G_1$  and  $G'$ , respectively, as

$$u_1 \triangleq \omega^{G_1} \cdot \mathbf{n}_3 \quad u_2 \triangleq \omega^{G'} \cdot \mathbf{n}_3 \quad (11)$$

where  $\mathbf{n}_3$  is one of three mutually perpendicular unit vectors directed as shown in Fig. 7.8.1, then the kinetic energy  $K$  of  $S$  is given by

$$K = \underset{(5.4.5)}{\frac{3}{2}Ju_1^2 + \frac{1}{2}J'u_2^2} \quad (12)$$

Consequently, the generalized momenta  $p_1$  and  $p_2$  are

$$p_1 = \underset{(4)}{\frac{\partial K}{\partial u_1}} \underset{(12)}{=} 3Ju_1 \quad p_2 = \underset{(4)}{\frac{\partial K}{\partial u_2}} \underset{(12)}{=} J'u_2 \quad (13)$$

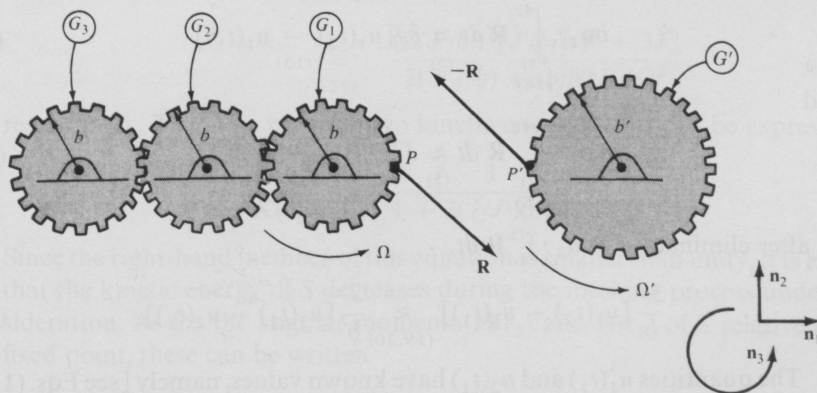


Figure 7.8.1

When  $G_1$  and  $G'$  are meshed suddenly, large forces come into play at the points  $P$  and  $P'$  (see Fig. 7.8.1) where  $G_1$  and  $G'$  come into contact with each other, as well as at the points of contact between  $G_1$  and  $G_2$ ,  $G_2$  and  $G_3$ , and each of  $G_1$ ,  $G_2$ ,  $G_3$ , and  $G'$  with its supports. Of all of these forces, the only ones that make contributions to generalized active forces (see Sec. 4.5), and hence to generalized impulses, are the interaction forces  $\mathbf{R}$  and  $-\mathbf{R}$  exerted on  $G_1$  at  $P$  and on  $G'$  at  $P'$ , respectively. The generalized impulses  $I_1$  and  $I_2$  are, therefore, given by

$$I_r = \underset{(1)}{\mathbf{v}_r^P} \cdot \int_{t_1}^{t_2} \mathbf{R} dt + \mathbf{v}_r^{P'} \cdot \int_{t_1}^{t_2} (-\mathbf{R}) dt \quad (r = 1, 2) \quad (14)$$

where the partial velocities  $\mathbf{v}_r^P$  and  $\mathbf{v}_r^{P'}$  ( $r = 1, 2$ ), found by inspection of the velocity expressions

$$\mathbf{v}^P = bu_1 \mathbf{n}_2 \quad \mathbf{v}^{P'} = -b'u_2 \mathbf{n}_2 \quad (15)$$

are

$$\mathbf{v}_1^P = b\mathbf{n}_2 \quad \mathbf{v}_1^{P'} = 0 \quad (16)$$

and

$$\mathbf{v}_2^P = 0 \quad \mathbf{v}_2^{P'} = -b'\mathbf{n}_2 \quad (17)$$

The symbols  $t_1$  and  $t_2$  in Eqs. (14) refer, respectively, to the instant of first contact of  $G_1$  and  $G'$  and to the instant at which the meshing process has been completed, and in writing Eqs. (14) it is presumed that the time interval  $t_2 - t_1$  is very small.

Using Eqs. (16) and (17) to form  $I_1$  and  $I_2$  in accordance with Eqs. (14), one finds that

$$I_1 = b\mathbf{n}_2 \cdot \int_{t_1}^{t_2} \mathbf{R} dt \quad I_2 = b'\mathbf{n}_2 \cdot \int_{t_1}^{t_2} \mathbf{R} dt \quad (18)$$

and substitution from Eqs. (13) and (18) into Eqs. (5) then yields

$$b\mathbf{n}_2 \cdot \int_{t_1}^{t_2} \mathbf{R} dt \underset{(18)}{\approx} \underset{(5)}{3J} \underset{(13)}{[u_1(t_2) - u_1(t_1)]} \quad (19)$$

and

$$b'\mathbf{n}_2 \cdot \int_{t_1}^{t_2} \mathbf{R} dt \underset{(18)}{\approx} \underset{(5)}{J'} \underset{(13)}{[u_2(t_2) - u_2(t_1)]} \quad (20)$$

or, after elimination of  $\mathbf{n}_2 \cdot \int_{t_1}^{t_2} \mathbf{R} dt$ ,

$$\frac{3J}{b} [u_1(t_2) - u_1(t_1)] \underset{(19,20)}{\approx} \frac{J'}{b'} [u_2(t_2) - u_2(t_1)] \quad (21)$$

The quantities  $u_1(t_1)$  and  $u_2(t_1)$  have known values, namely [see Eqs. (11)],

$$u_1(t_1) = 0 \quad u_2(t_1) = \Omega' \quad (22)$$

At time  $t_2$ , the meshing processing having been completed, points  $P$  and  $P'$  have equal velocities, which means that

$$bu_1(t_2) = -b'u_2(t_2) \quad (23)$$

(15)

or, if  $\Omega$  is defined as

$$\Omega \triangleq u_1(t_2) \quad (24)$$

that

$$u_2(t_2) = -\frac{b}{b'}u_1(t_2) = -\frac{b}{b'}\Omega \quad (25)$$

(23)                      (24)

Consequently,

$$\frac{3J}{b}(\Omega - 0) \approx \frac{J'}{b'}\left(-\frac{b}{b'}\Omega - \Omega'\right) \quad (26)$$

(24)    (22) (21)                      (25)                      (22)

and  $\Omega$ , the angular speed that was to be determined, is seen to be given by

$$\Omega = \frac{-(J'/b')\Omega'}{(26) \ 3(J/b) + (b/b')J'/b'} \quad (27)$$

Before leaving this example, we examine the changes that take place, during the time interval beginning at  $t_1$  and ending at  $t_2$ , in the kinetic energy of  $S$  and in the angular momentum of  $S$  with respect to any fixed point.

The kinetic energies of  $S$  at  $t_1$  and at  $t_2$  are given by

$$K(t_1) = \frac{1}{2}J'(\Omega')^2 \quad (28)$$

(12,22)

and

$$\begin{aligned} K(t_2) &= \frac{\Omega^2}{2} \left[ J' \left( \frac{b}{b'} \right)^2 + 3J \right] \\ &= \frac{(\Omega'J'/b')^2 [J'(b/b')^2 + 3J]}{2[3(J/b) + (b/b')J'/b']^2} \end{aligned} \quad (29)$$

(12,24,25)                      (27)

respectively. The ratio of these two kinetic energies thus can be expressed as

$$\frac{K(t_2)}{K(t_1)} = \frac{1}{1 + 3(J/J')(b'/b)^2} \quad (30)$$

(27)                      (28,29)

Since the right-hand member of this equation is smaller than unity, it is evident that the kinetic energy of  $S$  decreases during the meshing process under consideration. As for the angular momenta  $\mathbf{H}(t_1)$  and  $\mathbf{H}(t_2)$  of  $S$  relative to any fixed point, these can be written

$$\mathbf{H}(t_1) = J'\Omega'\mathbf{n}_3 \quad (31)$$

(22)

and

$$\mathbf{H}(t_2) = \left( -J' \frac{b}{b'} \Omega + J\Omega - J\Omega + J\Omega \right) \mathbf{n}_3 \quad (32)$$

(25)

where the last three terms in the parentheses can be seen to reflect the contributions of  $G_1$ ,  $G_2$ , and  $G_3$  when kinematical relationships are taken into account. From Eqs. (32) and (27) it thus follows that

$$\mathbf{H}(t_2) = - \frac{(\Omega'J'/b')(J - J'b/b')}{3(J/b) + (b/b')J'/b'} \mathbf{n}_3 \quad (33)$$

and the ratio of the magnitudes of  $\mathbf{H}(t_2)$  and  $\mathbf{H}(t_1)$  is

$$\frac{|\mathbf{H}(t_2)|}{|\mathbf{H}(t_1)|} = \frac{|1 - (J/J')(b'/b)|}{1 + 3(J/J')(b'/b)^2} \quad (34)$$

(33, 31)

Here, too, we have a quantity smaller than unity, so that angular momentum is seen to decrease also.

It is tempting to “explain” the decrease in kinetic energy by referring to the sound and heat generation known to accompany events of the kind under consideration. However, this is seen to be unsound when one realizes that the method at hand can also lead to kinetic energy increases (see, for example, Problem 14.6). What must be remembered is that Eqs. (5) are *approximate* relationships, which means that results obtained by using these equations can be somewhat, or even totally, unrealistic. Ultimately, only experiments can reveal the degree of utility of a given approximate solution of a problem. Hence, energy decreases should not be regarded as reassuring, or energy increases as alarming, in the present context. Most importantly, *one should not attempt to base a solution of a problem involving sudden velocity changes on an energy conservation principle.*

The change, if not the decrease, in angular momentum magnitude manifested in Eq. (34) can be explained readily. During the time interval beginning at  $t_1$  and ending at  $t_2$ , forces are exerted on  $S$  at points of contact between  $G'$  and its support and, similarly, on points of  $G_1$ ,  $G_2$ , and  $G_3$  by their supports. There is no reason to think that the sum of the moments of all of these forces about any fixed point vanishes. Hence, in accordance with the angular momentum principle [see Eq. (7.3.9)], the angular momentum of  $S$  with respect to any fixed point must be expected to change during the time interval beginning at  $t_1$  and ending at  $t_2$ . The principle of conservation of angular momentum applies to any system undergoing abrupt velocity changes, just as it applies to any system having other motions, only when the resultant moment about the system's mass center (or about a point fixed in a Newtonian reference frame) of all forces acting on the system is equal to zero.

## 7.9 COLLISIONS

When a system  $S$  is involved in a collision beginning at time  $t_1$  and ending at time  $t_2$ , the motion of  $S$  at time  $t_2$  frequently cannot be determined solely by use of Eqs. (7.8.5) together with a complete description of the motion of  $S$  at time  $t_1$ . Generally, some information about the velocity of one or more particles of  $S$  at time  $t_2$  must be used in addition to Eqs. (7.8.5), and this information must be expressed in mathematical form [see, for example, Eq. (7.8.23)]. What follows is an attempt to come to grips with this problem by formulating two assumptions that, as experiments have shown, are valid in many situations of practical interest.

In Fig. 7.9.1,  $P$  and  $P'$  designate points that come into contact with each other during a collision of two bodies  $B$  and  $B'$ . ( $P$  and  $P'$  are points of  $B$  and  $B'$ , respectively.)  $T$  is the plane that is tangent to the surfaces of  $B$  and  $B'$  at their point of contact, and  $\mathbf{n}$  is a unit vector perpendicular to  $T$ .

If  $\mathbf{v}^P(t)$  and  $\mathbf{v}^{P'}(t)$  denote the velocities of  $P$  and  $P'$ , respectively, at time  $t$ , then  $\mathbf{v}_A$ , defined as

$$\mathbf{v}_A \triangleq \mathbf{v}^P(t_1) - \mathbf{v}^{P'}(t_1) \quad (1)$$

is called the *velocity of approach* of  $B$  and  $B'$ , and  $\mathbf{v}_S$ , defined as

$$\mathbf{v}_S \triangleq \mathbf{v}^P(t_2) - \mathbf{v}^{P'}(t_2) \quad (2)$$

is called the *velocity of separation* of  $B$  and  $B'$ . Each of the vectors  $\mathbf{v}_A$  and  $\mathbf{v}_S$  can be resolved into two components, one parallel to  $\mathbf{n}$ , called the *normal component*, the other perpendicular to  $\mathbf{n}$ , called the *tangential component*, and the normal components can be expressed as  $\mathbf{n} \cdot \mathbf{v}_A \mathbf{n}$  and  $\mathbf{n} \cdot \mathbf{v}_S \mathbf{n}$ , while the tangential components are  $\mathbf{n} \times (\mathbf{v}_A \times \mathbf{n})$  and  $\mathbf{n} \times (\mathbf{v}_S \times \mathbf{n})$ . The velocity of approach, velocity of separation, and their normal and tangential components are shown in Fig. 7.9.2.

The first of the two aforementioned assumptions is that the normal components of  $\mathbf{v}_A$  and  $\mathbf{v}_S$  have opposite directions, while the magnitude of the normal component of  $\mathbf{v}_S$  is proportional to the magnitude of the normal component of  $\mathbf{v}_A$ , the constant

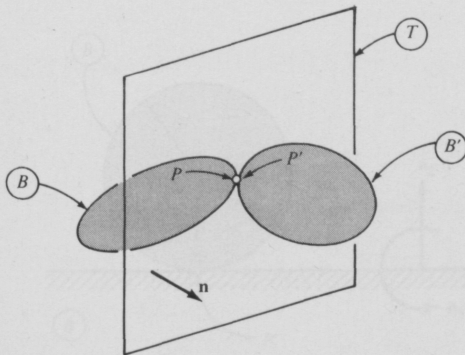


Figure 7.9.1

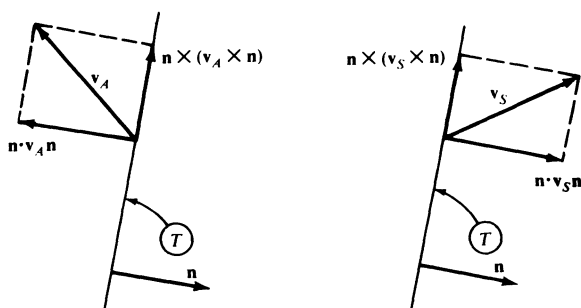


Figure 7.9.2

of proportionality being a quantity  $e$  whose value depends on material properties, but not on the motions, of  $B$  and  $B'$ . This can be stated analytically as (see Fig. 7.9.2)

$$\mathbf{n} \cdot \mathbf{v}_S = -e \mathbf{n} \cdot \mathbf{v}_A \quad (3)$$

The constant  $e$ , called a *coefficient of restitution*, is found, in practice, to take on values such that  $0 \leq e \leq 1$ . When  $e = 0$ , the collision is said to be *inelastic*;  $e = 1$  characterizes an idealized event termed a *perfectly elastic* collision.

The second assumption involves both the tangential component of  $\mathbf{v}_S$  (see Fig. 7.9.2) and the force  $\mathbf{R}$  exerted on  $B$  by  $B'$  at the point of contact between  $B$  and  $B'$  during the collision. More specifically,  $\mathbf{R}$  is integrated with respect to  $t$  from  $t_1$  to  $t_2$ ; the resulting vector is resolved into two components, one normal to the plane  $T$ , denoted by  $\mathbf{v}$ , and called the *normal impulse*, the other parallel to  $T$ , denoted by  $\boldsymbol{\tau}$ , and called the *tangential impulse*, as indicated in Fig. 7.9.3. The assumption is this: If and only if the inequality

$$|\boldsymbol{\tau}| < \mu |\mathbf{v}| \quad (4)$$

is satisfied, where  $\mu$  is the coefficient of static friction for  $B$  and  $B'$  (see Sec. 4.10), then there is no slipping at  $t_2$ , which means that (see Fig. 7.9.2)

$$\mathbf{n} \times (\mathbf{v}_S \times \mathbf{n}) = 0 \quad (5)$$

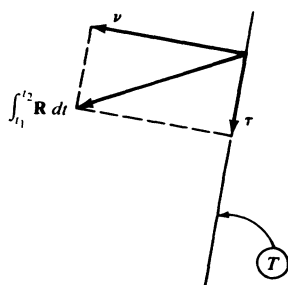


Figure 7.9.3



If the inequality (4) is violated, then

$$\tau = -\mu' |\mathbf{v}| \frac{\mathbf{n} \times (\mathbf{v}_S \times \mathbf{n})}{|\mathbf{n} \times (\mathbf{v}_S \times \mathbf{n})|} \quad (6)$$

where  $\mu'$  is the coefficient of kinetic friction for  $B$  and  $B'$  (see Sec. 4.10), and there is slipping at  $t_2$ , so that Eq. (5) does not apply.

**Example** Consider the collision of a uniform sphere  $B$  of mass  $m$  and radius  $b$  with a fixed body  $B'$  that is bounded by a horizontal plane, as indicated in Fig. 7.9.4. The angular velocity  $\boldsymbol{\omega}$  of  $B$  and the velocity  $\mathbf{v}^*$  of the center  $B^*$  of  $B$  can be expressed in terms of generalized speeds  $u_1, \dots, u_6$  as

$$\boldsymbol{\omega} = u_1 \mathbf{n}_1 + u_2 \mathbf{n}_2 + u_3 \mathbf{n}_3 \quad (7)$$

and

$$\mathbf{v}^* = u_4 \mathbf{n}_1 + u_5 \mathbf{n}_2 + u_6 \mathbf{n}_3 \quad (8)$$

The values of  $u_1, \dots, u_6$  at the instant  $t_1$  at which  $B$  comes into contact with  $B'$  are presumed to be known. It is desired to determine the values of  $u_1, \dots, u_6$  at time  $t_2$ , the instant at which  $B$  loses contact with  $B'$ .

The kinetic energy  $K$  of  $B$  is given by [see Eqs. (5.4.2), (5.4.4), and (5.4.7)]

$$K = \frac{1}{2} J (u_1^2 + u_2^2 + u_3^2) + \frac{1}{2} m (u_4^2 + u_5^2 + u_6^2) \quad (9)$$

where

$$J \triangleq \frac{2}{5} m b^2 \quad (10)$$

The generalized momenta  $p_1, \dots, p_6$ , formed in accordance with Eqs. (7.8.4), are thus

$$p_r = \begin{cases} J u_r & (r = 1, 2, 3) \\ m u_r & (r = 4, 5, 6) \end{cases} \quad (11)$$

The velocity of the point  $P$  of  $B$  that comes into contact with  $B'$  at time  $t_1$  is given at all times by

$$\mathbf{v}^P = \mathbf{v}^* + \boldsymbol{\omega} \times \mathbf{p} \quad (12)$$

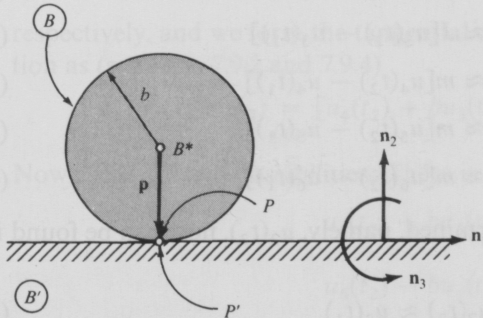


Figure 7.9.4

where  $\mathbf{p}$  is the position vector from  $B^*$  to  $P$ . Hence,

$$\mathbf{v}^P_{(12,7,8)} = u_4 \mathbf{n}_1 + u_5 \mathbf{n}_2 + u_6 \mathbf{n}_3 + (u_1 \mathbf{n}_1 + u_2 \mathbf{n}_2 + u_3 \mathbf{n}_3) \times \mathbf{p} \quad (13)$$

and the partial velocities of  $P$  at any time  $t$  are

$$\mathbf{v}_r^P(t) = \begin{cases} \mathbf{n}_r \times \mathbf{p} & (r = 1, 2, 3) \\ \mathbf{n}_{r-3} & (r = 4, 5, 6) \end{cases} \quad (14)$$

At time  $t_1$ ,  $\mathbf{p}$  is given by  $\mathbf{p} = -b\mathbf{n}_2$ , and the first three of Eqs. (14) thus lead to

$$\mathbf{v}_1^P(t_1) = -b\mathbf{n}_3 \quad \mathbf{v}_2^P(t_1) = 0 \quad \mathbf{v}_3^P(t_1) = b\mathbf{n}_1 \quad (15)$$

Letting  $\mathbf{R}$  be the force exerted on  $B$  by  $B'$  at their point of contact during the time interval beginning at  $t_1$  and ending at  $t_2$ , and defining  $S_i$  as

$$S_i \triangleq \mathbf{n}_i \cdot \int_{t_1}^{t_2} \mathbf{R} dt \quad (i = 1, 2, 3) \quad (16)$$

one can write

$$\int_{t_1}^{t_2} \mathbf{R} dt = \underset{(16)}{S_1 \mathbf{n}_1 + S_2 \mathbf{n}_2 + S_3 \mathbf{n}_3} \quad (17)$$

whereupon one is in position to express the generalized impulses  $I_1, \dots, I_6$  as

$$I_1 = \underset{(7.8.1)}{-b\mathbf{n}_3} \cdot \underset{(15)}{(S_1 \mathbf{n}_1 + S_2 \mathbf{n}_2 + S_3 \mathbf{n}_3)} = -bS_3 \quad (18)$$

$$I_2 = \underset{(7.8.1)}{0} \underset{(15,17)}{} \quad (19)$$

$$I_3 = \underset{(7.8.1)}{bS_1} \underset{(15,17)}{} \quad (20)$$

$$I_r = \underset{(7.8.1)}{S_{r-3}} \underset{(14,17)}{} \quad (r = 4, 5, 6) \quad (21)$$

so that substitution from Eqs. (11) and (18)–(21) into Eqs. (7.8.5) produces

$$-bS_3 \approx J[u_1(t_2) - u_1(t_1)] \quad (22)$$

$$0 \approx J[u_2(t_2) - u_2(t_1)] \quad (23)$$

$$bS_1 \approx J[u_3(t_2) - u_3(t_1)] \quad (24)$$

$$S_1 \approx m[u_4(t_2) - u_4(t_1)] \quad (25)$$

$$S_2 \approx m[u_5(t_2) - u_5(t_1)] \quad (26)$$

$$S_3 \approx m[u_6(t_2) - u_6(t_1)] \quad (27)$$

One of the quantities to be determined, namely,  $u_2(t_2)$ , now can be found immediately, for Eq. (23) yields

$$u_2(t_2) \approx u_2(t_1) \quad (28)$$

As for the rest, information in addition to that furnished by Eqs. (22) and (24)–(27) is required because these five equations involve the eight unknowns  $u_1(t_2)$ ,  $u_3(t_2)$ ,  $\dots$ ,  $u_6(t_2)$ ,  $S_1$ ,  $S_2$ , and  $S_3$ .

The velocity of the point  $P'$  of  $B'$  (see Fig. 7.9.4) with which  $B$  comes into contact during the collision is equal to zero at all times if  $B'$  is fixed, as is being presumed. Hence,

$$\mathbf{v}^{P'}(t_1) = \mathbf{v}^{P'}(t_2) = 0 \quad (29)$$

and  $\mathbf{v}_A$ , the velocity of approach, can be expressed as

$$\mathbf{v}_A = \underset{(1)}{\mathbf{v}^P(t_1)} - \underset{(29)}{0} \quad (30)$$

or, since  $\mathbf{p} = -b\mathbf{n}_2$  at  $t_1$ , as

$$\mathbf{v}_A = \underset{(30,13)}{[u_4(t_1) + bu_3(t_1)]\mathbf{n}_1 + u_5(t_1)\mathbf{n}_2 + [u_6(t_1) - bu_1(t_1)]\mathbf{n}_3} \quad (31)$$

Similarly,  $\mathbf{v}_S$ , the velocity of separation, formed in accordance with Eq. (2), is given by

$$\mathbf{v}_S = \underset{(2,29,13)}{[u_4(t_2) + bu_3(t_2)]\mathbf{n}_1 + u_5(t_2)\mathbf{n}_2 + [u_6(t_2) - bu_1(t_2)]\mathbf{n}_3} \quad (32)$$

and, with  $\mathbf{n}_2$  playing the role of  $\mathbf{n}$  (see Figs. 7.9.1 and 7.9.4), one can thus write one further equation by appealing to Eqs. (3), namely,

$$u_5(t_2) = \underset{(3,32,31)}{-eu_5(t_1)} \quad (33)$$

Furthermore, Eq. (26) may be replaced with

$$S_2 \approx -m(1 + e)u_5(t_1) \quad (34)$$

Four equations, namely, Eqs. (22), (24), (25), and (27), now are available for the determination of the remaining six unknowns,  $u_1(t_2)$ ,  $u_3(t_2)$ ,  $u_4(t_2)$ ,  $u_6(t_2)$ ,  $S_1$ , and  $S_3$ . To supplement these, we note that  $\mathbf{v}$ , the normal impulse, and  $\boldsymbol{\tau}$ , the tangential impulse, are given by [see Fig. 7.9.3, Eq. (17), and Fig. 7.9.4]

$$\mathbf{v} = S_2 \mathbf{n}_2 \quad (35)$$

and

$$\boldsymbol{\tau} = S_1 \mathbf{n}_1 + S_3 \mathbf{n}_3 \quad (36)$$

respectively, and we form the tangential component of the velocity of separation as (see Figs. 7.9.2 and 7.9.4)

$$\mathbf{n}_2 \times (\mathbf{v}_S \times \mathbf{n}_2) = \underset{(32)}{[u_4(t_2) + bu_3(t_2)]\mathbf{n}_1 + [u_6(t_2) - bu_1(t_2)]\mathbf{n}_3} \quad (37)$$

Now there are two possibilities: There is no slipping at  $t_2$ , in which case

$$u_4(t_2) + bu_3(t_2) = \underset{(5,37)}{0} \quad (38)$$

$$u_6(t_2) - bu_1(t_2) = \underset{(5,37)}{0} \quad (39)$$

and

$$(S_1^2 + S_3^2)^{1/2} < \mu |S_2| \quad (40)$$

(36) (4) (35)

Alternatively, there is slipping at  $t_2$ , and Eq. (6) requires that

$$S_1 = -\mu' |S_2| \frac{u_4(t_2) + bu_3(t_2)}{\{[u_4(t_2) + bu_3(t_2)]^2 + [u_6(t_2) - bu_1(t_2)]^2\}^{1/2}} \quad (41)$$

(36) (6) (35) (37)

and

$$S_3 = -\mu' |S_2| \frac{u_6(t_2) - bu_1(t_2)}{\{[u_4(t_2) + bu_3(t_2)]^2 + [u_6(t_2) - bu_1(t_2)]^2\}^{1/2}} \quad (42)$$

(36) (6) (35) (37)

We shall examine these two possibilities separately. Before doing so, however, we establish two relationships that apply in both cases. Specifically, we eliminate  $S_1$  from Eqs. (24) and (25), obtaining

$$bm[u_4(t_2) - u_4(t_1)] \approx J[u_3(t_2) - u_3(t_1)] \quad (43)$$

Similarly, eliminating  $S_3$  from Eqs. (22) and (27), we find that

$$-bm[u_6(t_2) - u_6(t_1)] \approx J[u_1(t_2) - u_1(t_1)] \quad (44)$$

If there is no slipping at  $t_2$ , then Eq. (38) applies, and elimination of  $u_4(t_2)$  from Eqs. (38) and (43) reveals that

$$u_3(t_2) \approx \frac{Ju_3(t_1) - mbu_4(t_1)}{mb^2 + J} \quad (45)$$

Once  $u_3(t_2)$  has been evaluated, one can find  $u_4(t_2)$  by using the relationship

$$u_4(t_2) = -bu_3(t_2) \quad (46)$$

(38)

As for  $u_1(t_2)$ , elimination of  $u_6(t_2)$  from Eqs. (39) and (44) results in

$$u_1(t_2) \approx \frac{Ju_1(t_1) + mbu_6(t_1)}{mb^2 + J} \quad (47)$$

and Eq. (39) then permits one to evaluate  $u_6(t_2)$  as

$$u_6(t_2) = bu_1(t_2) \quad (48)$$

Successive use of Eqs. (28), (33), and (45)–(48) thus yields a set of values of  $u_1, \dots, u_6$  at time  $t_2$ , and these values apply if and only if the inequality (40) is satisfied when  $S_1, S_2$ , and  $S_3$  have the values given by Eqs. (25), (26), and (27), respectively.

If there is slipping at  $t_2$ , then, as before,  $u_2(t_2)$  and  $u_5(t_2)$  are given by Eqs. (28) and (33), respectively, and  $S_2$  can be found with the aid of Eq. (26).

To determine  $u_1(t_2)$ ,  $u_3(t_2)$ ,  $u_4(t_2)$ , and  $u_6(t_2)$ , one can begin by using Eqs. (25) and (43) to express  $S_1$  as

$$S_1 \approx \frac{J}{b} [u_3(t_2) - u_3(t_1)] \quad (49)$$

and referring to Eqs. (27) and (44) to obtain

$$S_3 \approx -\frac{J}{b} [u_1(t_2) - u_1(t_1)] \quad (50)$$

Next,

$$u_3(t_2) \underset{(49)}{\approx} u_3(t_1) + \frac{bS_1}{J} \quad (51)$$

$$u_1(t_2) \underset{(50)}{\approx} u_1(t_1) - \frac{bS_3}{J} \quad (52)$$

$$u_4(t_2) \underset{(43)}{\approx} u_4(t_1) + \frac{S_1}{m} \underset{(51)}{\quad} \quad (53)$$

$$u_6(t_2) \underset{(44)}{\approx} u_6(t_1) + \frac{S_3}{m} \underset{(52)}{\quad} \quad (54)$$

Hence,

$$u_4(t_2) + bu_3(t_2) \underset{(51,53)}{\approx} u_4(t_1) + bu_3(t_1) + \left(\frac{1}{m} + \frac{b^2}{J}\right)S_1 \quad (55)$$

$$u_6(t_2) - bu_1(t_2) \underset{(52,54)}{\approx} u_6(t_1) - bu_1(t_1) + \left(\frac{1}{m} + \frac{b^2}{J}\right)S_3 \quad (56)$$

and, if  $\alpha$ ,  $\gamma$ , and  $k$  are defined as

$$\alpha \triangleq u_4(t_1) + bu_3(t_1) \quad \gamma \triangleq u_6(t_1) - bu_1(t_1) \quad (57)$$

and

$$k \triangleq \frac{1}{m} + \frac{b^2}{J} \quad (58)$$

respectively, then Eqs. (41) and (42) lead to

$$S_1 \underset{(41)}{\approx} -\mu'|S_2| \frac{\alpha + kS_1}{[(\alpha + kS_1)^2 + (\gamma + kS_3)^2]^{1/2}} \underset{(55-58)}{\quad} \quad (59)$$

and

$$S_3 \underset{(42)}{\approx} -\mu'|S_2| \frac{\gamma + kS_3}{[(\alpha + kS_1)^2 + (\gamma + kS_3)^2]^{1/2}} \underset{(55-58)}{\quad} \quad (60)$$

from which it follows that

$$\frac{S_1}{S_3} \underset{(59,60)}{\approx} \frac{\alpha + kS_1}{\gamma + kS_3} \quad (61)$$

or, equivalently, that

$$S_3 \approx \frac{\gamma}{\alpha} S_1 \quad (62)$$

which makes it possible to replace Eq. (59) with

$$\begin{aligned} S_1 &\underset{(59)}{\approx} -\mu'|S_2| \frac{\alpha + kS_1}{\{(\alpha + kS_1)^2 + [\gamma + k(\gamma/\alpha)S_1]^2\}^{1/2}} \\ &= -\mu'|S_2| \frac{\alpha + kS_1}{|\alpha + kS_1|[1 + (\gamma/\alpha)^2]^{1/2}} \end{aligned} \quad (63)$$

Now,  $(\alpha + kS_1)/|\alpha + kS_1|$  has the value 1 when  $\alpha + kS_1 > 0$ , and the value  $-1$  when  $\alpha + kS_1 < 0$ . In the first case, therefore,

$$S_1 \underset{(63)}{\approx} -\frac{\mu'|S_2|}{[1 + (\gamma/\alpha)^2]^{1/2}} \quad (64)$$

so that

$$\alpha + kS_1 \underset{(64)}{\approx} \alpha - \frac{k\mu'|S_2|}{[1 + (\gamma/\alpha)^2]^{1/2}} > 0 \quad (65)$$

which implies that

$$\alpha > 0 \quad (66)$$

In the second case,

$$S_1 \underset{(63)}{\approx} \frac{\mu'|S_2|}{[1 + (\gamma/\alpha)^2]^{1/2}} \quad (67)$$

so that

$$\alpha + kS_1 \underset{(67)}{\approx} \alpha + \frac{k\mu'|S_2|}{[1 + (\gamma/\alpha)^2]^{1/2}} < 0 \quad (68)$$

which implies that

$$\alpha < 0 \quad (69)$$

Since the inequalities (66) and (69) are mutually exclusive, the sign of  $\alpha$  is sufficient to settle the question of whether Eq. (64) or Eq. (67) should be used,

and one can accommodate both equations by writing

$$S_1 \approx -\frac{\mu'\alpha|S_2|}{|\alpha|[1 + (\gamma/\alpha)^2]^{1/2}} \quad (70)$$

which brings one into position to evaluate  $S_3$  by reference to Eq. (62), whereupon one can find  $u_1(t_2)$ ,  $u_3(t_2)$ ,  $u_4(t_2)$ , and  $u_6(t_2)$  with the aid of Eqs. (52), (51), (53), and (54), respectively.

In Fig. 7.9.5, a complete algorithm for the evaluation of  $u_1, \dots, u_6$  at time  $t_2$  is set forth in the form of a flowchart, in which numbers in parentheses refer to corresponding equations. The values of the physical parameters  $b, m, J, e, \mu$ , and  $\mu'$ , and the generalized speeds  $u_1, \dots, u_6$  at time  $t_1$  are presumed to be known, and  $u_5(t_1)$  must be negative, since no collision will occur otherwise. The physical significance of the preceding analysis is brought to light with the aid of a numerical example.

Suppose that  $e = 0.8$ ,  $\mu = 0.25$ ,  $\mu' = 0.20$ , and  $B$  has a “topspin” when it strikes  $B'$ , which is the case, for instance, if [see Eqs. (7) and (8) and Fig. 7.9.4]

$$u_1(t_1) = u_2(t_1) = 0 \quad u_3(t_1) = -\frac{3V}{b} \quad (71)$$

$$u_4(t_1) = -u_5(t_1) = V \quad u_6(t_1) = 0 \quad (72)$$

where  $V$  is any (positive) speed. Then, following the steps indicated in Fig. 7.9.5, one finds that  $S_1 \approx 4mV/7$ ,  $S_2 \approx 1.8mV$ ,  $S_3 \approx 0$ . Hence, the inequality

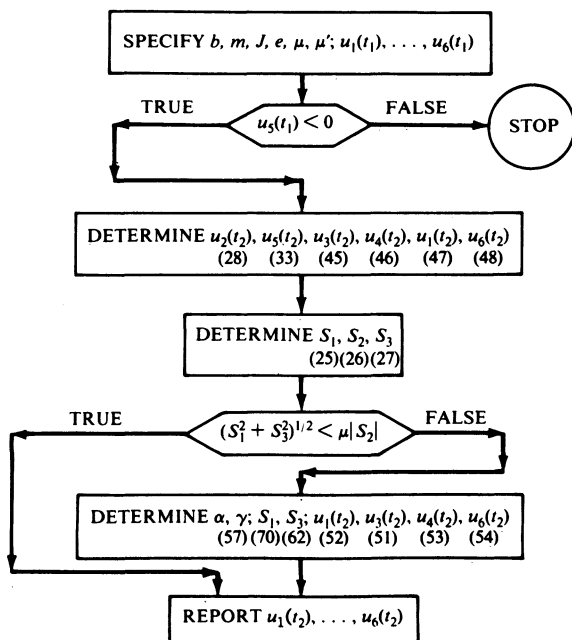


Figure 7.9.5

(40) is violated, which means that there is slipping at  $t_2$ . Proceeding in accordance with Fig. 7.9.5, one obtains

$$u_1(t_2) = u_2(t_2) = 0 \quad u_3(t_2) = -\frac{2.1V}{b} \quad (73)$$

$$u_4(t_2) = 1.36V \quad u_5(t_2) = 0.8V \quad u_6(t_2) = 0 \quad (74)$$

The fact that  $u_6(t_2) = 0$  shows that  $B^*$  moves in the same plane subsequent to the collision of  $B$  with  $B'$  as it does prior to the collision. The angles  $\theta_1$  and  $\theta_2$  that the vector  $\mathbf{v}^*$  [see Eq. (8)] makes with the vertical before and after impact, respectively, can be compared with each other when it is noted that

$$\theta_i = \tan^{-1} \left| \frac{u_4(t_i)}{u_5(t_i)} \right| \quad (i = 1, 2) \quad (75)$$

so that [see Eqs. (72)]  $\theta_1 = 45^\circ$ , while [see Eqs. (74)]  $\theta_2 = 59.5^\circ$ . This means that the topspin initially imparted to  $B$  has the effect of producing a "drop," a fact that will not surprise tennis players.

Finally, one can estimate how long  $B$  will continue to bounce. The maximum height reached by  $B^*$  during any bounce is, of course, smaller than that attained during the preceding one, so that this height approaches zero (and bouncing ceases) when the number  $n$  of bounces approaches infinity. The time  $T_n$  required for  $n$  bounces to occur is found by noting that the time that elapses between two successive impacts depends only on the value of  $u_5$  at the end of the first of these; that is, if  $\tau_n$  denotes the time required for the  $n$ th bounce, then  $\tau_1$  is given by

$$\tau_1 = \frac{2u_5(t_2)}{g} \underset{(33)}{=} \frac{-2eu_5(t_1)}{g} \quad (76)$$

and, similarly,

$$\tau_2 = \frac{2u_5(t_4)}{g} = \frac{-2eu_5(t_3)}{g} \quad (77)$$

But

$$u_5(t_3) = -u_5(t_2) \underset{(33)}{=} eu_5(t_1) \quad (78)$$

Hence,

$$\tau_2 \underset{(77,78)}{=} \frac{-2e^2u_5(t_1)}{g} \quad (79)$$

Similarly,

$$\tau_3 = \frac{-2e^3u_5(t_1)}{g} \quad (80)$$



and

$$\tau_n = \frac{-2e^n u_5(t_1)}{g} \quad (81)$$

Consequently,

$$T_n = \tau_1 + \cdots + \tau_n = \frac{-2(e + e^2 + \cdots + e^n)u_5(t_1)}{g} \quad (82)$$

Now,

$$e + e^2 + \cdots + e^n = \frac{e(1 - e^n)}{1 - e} \quad (83)$$

Thus,

$$T_n \underset{(82,83)}{=} \frac{-2e(1 - e^n)u_5(t_1)}{(1 - e)g} \quad (84)$$

and the total time  $T$  required for infinitely many bounces is given by (for  $0 < e < 1$ )

$$T = \lim_{n \rightarrow \infty} T_n \underset{(84)}{=} \frac{-2eu_5(t_1)}{(1 - e)g} \quad (85)$$

Since this analysis does not account for the time consumed by the collisions (infinitely many), Eq. (85) should be regarded as a lower bound on the time required for bouncing to cease subsequent to time  $t_1$ . Applied to the numerical example considered previously, Eq. (85) yields

$$T = \frac{-2(0.8)(-V)}{(1 - 0.8)g} = \frac{8V}{g} \quad (86)$$

Hence, if  $V = 10$  m/s,  $B$  may be expected to bounce for a little longer than 8 s.

## 7.10 MOTIONS GOVERNED BY LINEAR DIFFERENTIAL EQUATIONS

Occasionally, one encounters a system whose motion is governed by one or more *linear* differential equations. More frequently, linear differential equations arise as a result of linearizations performed to take advantage of the fact that one can solve certain kinds of linear differential equations without resorting to numerical integration. The analyses dealing with the Foucault pendulum in Secs. 6.1 and 6.4 furnish cases in point. Because only one of Eqs. (6.1.29), (6.1.30), and (6.1.34) is a linear differential equation [the second of Eqs. (6.1.34)], this set of equations was integrated numerically to obtain the results reported in Table 6.1.1; but Eqs. (6.4.6)–(6.4.8), being linear differential equations, were solved in closed form.

One linear differential equation that arises frequently both in dynamics and in other areas of physics has the form

$$\ddot{x} + 2np\dot{x} + p^2x = f(t) \quad (1)$$

where  $x$  is a function of time  $t$ , dots denote time-differentiation,  $n$  and  $p$  are constants called, respectively, the *fraction of critical damping* and the *circular natural frequency*, and  $f(t)$  is a specified function of  $t$ , called the *forcing function*. [See, for example, Eq. (7.4.21), where  $u_3$  plays the role of  $x$  while  $n = f(t) = 0$ .] The general solution of Eq. (1) is

$$x = \frac{1}{a_1 - a_2} \left\{ [\dot{x}(0) - a_2 x(0)] e^{a_1 t} - [\dot{x}(0) - a_1 x(0)] e^{a_2 t} + \int_0^t f(\zeta) [e^{a_1(t-\zeta)} - e^{a_2(t-\zeta)}] d\zeta \right\} \quad (2)$$

where  $x(0)$  and  $\dot{x}(0)$  denote, respectively, the values of  $x$  and  $\dot{x}$  at  $t = 0$ , and  $a_1$  and  $a_2$  are defined as

$$a_1 \triangleq -p[n - (n^2 - 1)^{1/2}] \quad a_2 \triangleq -p[n + (n^2 - 1)^{1/2}] \quad (3)$$

The set of  $n$  linear differential equations

$$\sum_{s=1}^n (M_{rs}\ddot{q}_s + K_{rs}q_s) = 0 \quad (r = 1, \dots, n) \quad (4)$$

governs motions of a holonomic system  $S$  under conditions stated in Problem 12.4. Given the constants  $M_{rs}$  and  $K_{rs}$  ( $r, s = 1, \dots, n$ ), as well as  $q_r(0)$  and  $\dot{q}_r(0)$ , the initial values of  $q_r$  and  $\dot{q}_r$  ( $r = 1, \dots, n$ ), respectively, one can find  $q_1, \dots, q_n$  for  $t > 0$  by proceeding as follows.

Let  $M$  and  $K$  be  $n \times n$  matrices respectively having  $M_{rs}$  and  $K_{rs}$  as the elements in row  $r$ , column  $s$ .  $M$  and  $K$  are called the *mass matrix* and the *stiffness matrix*, respectively.

Construct an upper-triangular  $n \times n$  matrix  $V$  such that

$$M = V^T V \quad (5)$$

where  $V^T$  denotes the transpose of  $V$ . (When  $M$  is diagonal, then  $V = M^{1/2}$ , that is,  $V$  is the diagonal matrix whose elements are the square roots of corresponding elements of  $M$ .)

Find the eigenvalues  $\lambda_1, \dots, \lambda_n$  of the (necessarily symmetric)  $n \times n$  matrix  $W$  defined as

$$W \triangleq (V^{-1})^T K V^{-1} \quad (6)$$

and determine the corresponding eigenvectors (each an  $n \times 1$  matrix)  $C^{(1)}, \dots, C^{(n)}$  of  $W$ . If not all eigenvalues of  $W$  are distinct, then, corresponding to an eigenvalue of multiplicity  $\mu$ , construct  $\mu$  eigenvectors that are orthogonal to each other.†

† The Gram-Schmidt orthogonalization procedure may be used to accomplish this task.

Define an  $n \times 1$  matrix  $B^{(i)}$ , a scalar  $N_i$ , and an  $n \times 1$  matrix  $A^{(i)}$  ( $i = 1, \dots, n$ ) as, respectively,

$$B^{(i)} \triangleq V^{-1}C^{(i)} \quad (i = 1, \dots, n) \quad (7)$$

$$N_i \triangleq \sqrt{[B^{(i)}]^T M B^{(i)}} \quad (i = 1, \dots, n) \quad (8)$$

and

$$A^{(i)} \triangleq \frac{B^{(i)}}{N_i} \quad (i = 1, \dots, n) \quad (9)$$

and let  $A$  be the  $n \times n$  matrix having  $A^{(1)}, \dots, A^{(n)}$  as columns; that is, define  $A$  as

$$A \triangleq [A^{(1)}; A^{(2)}; \dots; A^{(n)}] \quad (10)$$

The matrix  $A$  is called the *modal matrix normalized with respect to the mass matrix*.

Let  $q(0)$  and  $\dot{q}(0)$  be  $n \times 1$  matrices respectively having  $q_r(0)$  and  $\dot{q}_r(0)$  as the elements in the  $r$ th row, and define  $Q(0)$  and  $\dot{Q}(0)$  as the  $n \times 1$  matrices

$$Q(0) \triangleq A^T M q(0) \quad \dot{Q}(0) \triangleq A^T M \dot{q}(0) \quad (11)$$

Introduce scalars  $p_1, \dots, p_n$  as

$$p_r \triangleq \lambda_r^{1/2} \quad (r = 1, \dots, n) \quad (12)$$

and define diagonal  $n \times n$  matrices  $c$ ,  $s$ , and  $p$  as

$$c \triangleq \begin{bmatrix} \cos p_1 t & 0 & \dots & 0 \\ 0 & \cos p_2 t & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \cos p_n t \end{bmatrix} \quad (13)$$

$$s \triangleq \begin{bmatrix} \sin p_1 t & 0 & \dots & 0 \\ 0 & \sin p_2 t & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sin p_n t \end{bmatrix} \quad (14)$$

and

$$p \triangleq \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n \end{bmatrix} \quad (15)$$

Finally, form the  $n \times 1$  matrix  $Q$  defined as

$$Q \triangleq cQ(0) + p^{-1}s\dot{Q}(0) \quad (16)$$

and let  $q$  be the  $n \times 1$  matrix whose elements are the generalized coordinates  $q_1, \dots, q_n$ . Then  $q$  is given by

$$q = AQ \quad (17)$$

The elements  $Q_1, \dots, Q_n$  of  $Q$  are called *normal coordinates*. The system  $S$  can move in such a way that any one of the normal coordinates, say, the  $j$ th, varies with time while all of the rest of the normal coordinates vanish. This occurs when  $Q_j(0)$  and/or  $\dot{Q}_j(0)$  differ from zero, but  $Q_k(0)$  and  $\dot{Q}_k(0)$  vanish for  $k \neq j$  ( $k = 1, \dots, n$ ), for then  $Q_j$  is the only nonvanishing element of  $Q$ , as can be seen by reference to Eqs. (13)–(16). Under these circumstances, the expression for  $q$  produced by Eq. (17) involves only the  $j$ th column of the modal matrix  $A$ ; that is,

$$q = A^{(j)}Q_j \quad (18)$$

and  $S$  is said to be *moving in the  $j$ th normal mode*. To find initial conditions such that  $S$  moves in this manner, one needs only to assign arbitrary values to  $Q_j(0)$  and  $\dot{Q}_j(0)$ , set  $Q_k(0)$  and  $\dot{Q}_k(0)$  equal to zero for  $k \neq j$  ( $k = 1, \dots, n$ ), and assign to  $q(0)$  and  $\dot{q}(0)$  the values  $A^{(j)}Q_j(0)$  and  $A^{(j)}\dot{Q}_j(0)$ , respectively.

Equation (17) is equivalent to the set of equations

$$q_r = \sum_{s=1}^n A_r^{(s)}Q_s \quad (r = 1, \dots, n) \quad (19)$$

which show that  $q_r$  ( $r = 1, \dots, n$ ) can be regarded as a sum each of whose elements represents a motion of  $S$  in a normal mode [see Eq. (18)]. Moreover, Eqs. (19) furnish the basis for the constructing of *approximations* to  $q_1, \dots, q_n$  by *modal truncation*, that is, by letting  $v$  be an integer smaller than  $n$  and writing

$$q_r \approx \sum_{s=1}^v A_r^{(s)}Q_s \quad (r = 1, \dots, n) \quad (20)$$

or, equivalently,

$$q \approx \tilde{A}\tilde{Q} \quad (21)$$

where  $\tilde{A}$  is the  $n \times v$  matrix whose columns are the first  $v$  columns of  $A$ , and  $\tilde{Q}$  is the  $v \times 1$  matrix whose elements are the first  $v$  elements of  $Q$ . Such an approximation can save a considerable amount of computational effort when  $v$  is sufficiently small in comparison with  $n$ , for one needs only the  $v$  eigenvalues  $\lambda_1, \dots, \lambda_v$  and the  $v$  eigenvectors  $C^{(1)}, \dots, C^{(v)}$ , rather than all  $n$  eigenvalues and eigenvectors, to evaluate  $\tilde{A}$  and  $\tilde{Q}$ .

If  $F_1(t), \dots, F_n(t)$  are known functions of  $t$ , and  $q_1, \dots, q_n$  are governed by

$$\sum_{s=1}^n (M_{rs}\ddot{q}_s + K_{rs}q_s) = F_r(t) \quad (r = 1, \dots, n) \quad (22)$$

rather than by Eqs. (4), but  $M_{rs}$  and  $K_{rs}$  ( $r, s = 1, \dots, n$ ) have the same meanings as heretofore, then  $q$  is again given by Eq. (17) if  $Q$ , rather than being defined as in Eq. (16), is taken to be

$$Q \triangleq cQ(0) + p^{-1}[s\dot{Q}(0) + \eta] \quad (23)$$

where  $\eta$  is an  $n \times 1$  matrix whose  $r$ th element is defined as

$$\eta_r \triangleq \sum_{j=1}^n A_j^{(r)} [\gamma_j^{(r)} \sin p_r t - \sigma_j^{(r)} \cos p_r t] \quad (r = 1, \dots, n) \quad (24)$$

with

$$\gamma_j^{(r)} \triangleq \int_0^t F_j(\zeta) \cos p_r \zeta d\zeta \quad \sigma_j^{(r)} \triangleq \int_0^t F_j(\zeta) \sin p_r \zeta d\zeta \quad (j, r = 1, \dots, n) \quad (25)$$

Furthermore, approximate solutions of Eqs. (22) can be obtained by modal truncation, that is, by using Eq. (21) and taking for  $\tilde{Q}$  the  $\nu \times 1$  matrix whose elements are the first  $\nu$  elements of  $Q$  as given by Eq. (23).

When  $n$  is sufficiently small, say, less than 4, one can solve Eqs. (4) and (22) by methods simpler than the ones just set forth. Conversely, when  $n$  is relatively large, the use of normal modes is very effective, especially when good computer programs for performing matrix operations are readily available.

**Derivations** Differentiation of Eq. (2) with respect to  $t$  yields

$$\begin{aligned} \dot{x} = \frac{1}{a_1 - a_2} & \left\{ [\dot{x}(0) - a_2 x(0)] a_1 e^{a_1 t} - [\dot{x}(0) - a_1 x(0)] a_2 e^{a_2 t} \right. \\ & \left. + \int_0^t f(\zeta) [a_1 e^{a_1(t-\zeta)} - a_2 e^{a_2(t-\zeta)}] d\zeta \right\} \end{aligned} \quad (26)$$

and differentiating this equation one obtains

$$\begin{aligned} \ddot{x} = \frac{1}{a_1 - a_2} & \left\{ [\dot{x}(0) - a_2 x(0)] a_1^2 e^{a_1 t} - [\dot{x}(0) - a_1 x(0)] a_2^2 e^{a_2 t} \right. \\ & \left. + \int_0^t f(\zeta) [a_1^2 e^{a_1(t-\zeta)} - a_2^2 e^{a_2(t-\zeta)}] d\zeta \right\} + f(t) \end{aligned} \quad (27)$$

Using Eqs. (2), (26), and (27), one thus finds that

$$\begin{aligned} \ddot{x} + 2np\dot{x} + p^2x = \frac{1}{a_1 - a_2} & \left\langle \left\{ [\dot{x}(0) - a_2 x(0)] e^{a_1 t} \right. \right. \\ & \left. + \int_0^t f(\zeta) e^{a_1(t-\zeta)} d\zeta \right\} (a_1^2 + 2npa_1 + p^2) \\ & - \left\{ [\dot{x}(0) - a_1 x(0)] e^{a_2 t} \right. \\ & \left. + \int_0^t f(\zeta) e^{a_2(t-\zeta)} d\zeta \right\} (a_2^2 + 2npa_2 + p^2) \right\rangle + f(t) \end{aligned} \quad (28)$$

Now, if  $a_1$  and  $a_2$  are given by Eqs. (3), then

$$a_1^2 + 2npa_1 + p^2 = a_2^2 + 2npa_2 + p^2 = 0 \quad (29)$$

Consequently,

$$\ddot{x} + 2np\dot{x} + p^2x = f(t) \quad (30)$$

(28,29)

which is Eq. (1). Since  $x$  as given in Eq. (2) contains two arbitrary constants, it is thus seen to be the general solution of Eq. (1) when  $a_1$  and  $a_2$  are given by Eqs. (3).

To establish the validity of the procedure for finding functions  $q_1, \dots, q_n$  of  $t$  that satisfy Eqs. (4), we begin with some observations regarding the matrices  $M, K, C^{(i)}, B^{(i)}$ , and  $A^{(i)}$  ( $i = 1, \dots, n$ ), noting first that

$$[A^{(i)}]^T M A^{(i)} = \frac{[B^{(i)}]^T}{N_i} M \frac{B^{(i)}}{N_i} = 1 \quad (i = 1, \dots, n) \quad (31)$$

(9) (8)

Second, if  $B^{(i)}$  is the eigenvector of  $M^{-1}K$  corresponding to the eigenvalue  $\lambda_i$ , that is,  $M^{-1}KB^{(i)} = \lambda_i B^{(i)}$ , and  $C^{(i)}$  is the eigenvector of  $W$  corresponding to the eigenvalue  $\bar{\lambda}_i$ , that is,  $WC^{(i)} = \bar{\lambda}_i C^{(i)}$ , then it follows from Eq. (5) that  $V^{-1}(V^{-1})^T KB^{(i)} = \lambda_i B^{(i)}$  or, when Eqs. (7) are taken into account, that  $V^{-1}(V^{-1})^T K V^{-1}C^{(i)} = \lambda_i V^{-1}C^{(i)}$ , while Eq. (6) enables one to write  $(V^{-1})^T K V^{-1}C^{(i)} = \bar{\lambda}_i C^{(i)}$  or, after premultiplication with  $V^{-1}$ ,  $V^{-1}(V^{-1})^T K V^{-1}C^{(i)} = \bar{\lambda}_i V^{-1}C^{(i)}$  ( $i = 1, \dots, n$ ). Consequently,  $\lambda_i = \bar{\lambda}_i$  ( $i = 1, \dots, n$ ), which is to say that  $M^{-1}K$  and  $W$  have the same eigenvalues  $\lambda_1, \dots, \lambda_n$  and that  $B^{(i)}$  is the eigenvector of  $M^{-1}K$  corresponding to  $\lambda_i$  if  $B^{(i)}$  is related to  $C^{(i)}$ , the eigenvector of  $W$  corresponding to  $\lambda_i$  ( $i = 1, \dots, n$ ), as in Eqs. (7). Hence, we can write

$$M^{-1}KB^{(i)} = \lambda_i B^{(i)} \quad (i = 1, \dots, n) \quad (32)$$

or, after premultiplication with  $M$ ,

$$KB^{(i)} = \lambda_i MB^{(i)} \quad (i = 1, \dots, n) \quad (33)$$

(32)

Consequently,

$$\begin{aligned} [B^{(i)}]^T KB^{(i)} &= \lambda_i [B^{(i)}]^T MB^{(i)} \\ &= \lambda_i N_i^2 \quad (i = 1, \dots, n) \end{aligned} \quad (34)$$

(8)

and

$$\frac{[B^{(i)}]^T}{N_i} K \frac{B^{(i)}}{N_i} = \lambda_i \quad (35)$$

(34)

or, in view of Eqs. (9),

$$[A^{(i)}]^T KA^{(i)} = \lambda_i \quad (i = 1, \dots, n) \quad (36)$$

Third, we establish the validity of the *orthogonality* relationships

$$[A^{(j)}]^T MA^{(i)} = 0 \quad (i, j = 1, \dots, n; i \neq j) \quad (37)$$

and

$$[A^{(j)}]^T K A^{(i)} = 0 \quad (i, j = 1, \dots, n; i \neq j) \quad (38)$$

as follows.

Because  $W$  is symmetric, the eigenvectors of  $W$  are orthogonal to each other if the eigenvalues of  $W$  are distinct, and infinitely many sets of orthogonal eigenvectors of  $W$  can be found if not all eigenvalues are distinct. Hence,

$$[C^{(j)}]^T C^{(i)} = 0 \quad (i, j = 1, \dots, n; i \neq j) \quad (39)$$

Now,

$$\begin{aligned} [B^{(j)}]^T M B^{(i)} &= [V^{-1} C^{(j)}]^T M [V^{-1} C^{(i)}] \\ &\quad (7) \quad (7) \\ &= [C^{(j)}]^T [V^T]^{-1} M V^{-1} C^{(i)} \\ &= [C^{(j)}]^T [V^T]^{-1} V^T V V^{-1} C^{(i)} \\ &\quad (5) \\ &= [C^{(j)}]^T C^{(i)} = 0 \quad (i, j = 1, \dots, n; i \neq j) \quad (40) \\ &\quad (39) \end{aligned}$$

and Eqs. (37) follow directly from these equations together with Eqs. (9). Furthermore,

$$[B^{(j)}]^T K B^{(i)} = \lambda_i [B^{(j)}]^T M B^{(i)} = 0 \quad (i, j = 1, \dots, n; i \neq j) \quad (41)$$

(33) (40)

and the validity of Eqs. (38) thus is established when Eqs. (9) are taken into account.

As will be shown next, what is of interest in connection with Eqs. (4) is that Eqs. (31) and (37) imply that

$$A^T M A = U \quad (42)$$

while Eqs. (36) and (38) justify the conclusion that

$$A^T K A = \lambda \quad (43)$$

where  $A$  is the modal matrix defined in Eq. (10),  $U$  is the  $n \times n$  unit matrix, and  $\lambda$  is the  $n \times n$  diagonal matrix defined as

$$\lambda \triangleq \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad (44)$$

Specifically, Eqs. (4) are equivalent to the matrix differential equation

$$M\ddot{q} + Kq = 0 \quad (45)$$

so that, if  $q$  is set equal to  $AQ$  [see Eq. (17)], where  $Q$  is an, as yet, unknown  $n \times 1$  matrix whose elements are functions of  $t$ , then

$$MA\ddot{Q} + KAQ = 0 \quad (46)$$

(45)

and, premultiplying this equation with  $A^T$ , we find in view of Eqs. (42) and (43) that

$$\ddot{Q} + \lambda Q = 0 \quad (47)$$

or, since  $\lambda$  is a diagonal matrix [see Eq. (44)], that

$$\ddot{Q}_r + \underset{(12)}{p_r^2} Q_r = 0 \quad (r = 1, \dots, n) \quad (48)$$

The general solution of these equations is

$$Q_r = Q_r(0) \cos p_r t + \frac{\dot{Q}_r(0)}{p_r} \sin p_r t \quad (r = 1, \dots, n) \quad (49)$$

where  $Q_r(0)$  and  $\dot{Q}_r(0)$  are, respectively, the initial values of  $Q_r$  and  $\dot{Q}_r$  ( $r = 1, \dots, n$ ). Moreover, if  $Q(0)$  and  $\dot{Q}(0)$  are the  $n \times 1$  matrices whose elements are, respectively,  $Q_1(0), \dots, Q_n(0)$  and  $\dot{Q}_1(0), \dots, \dot{Q}_n(0)$ , and  $c$ ,  $s$ , and  $p$  are the matrices defined in Eqs. (13)–(15), then Eq. (16) is equivalent to Eqs. (49). What remains to be shown is that  $Q(0)$  and  $\dot{Q}(0)$  as defined in Eqs. (11) are, in fact, the initial values of  $Q$  and  $\dot{Q}$ , respectively. This is accomplished by referring to Eq. (17) to write

$$q(0) = A Q(0) \quad \dot{q}(0) = A \dot{Q}(0) \quad (50)$$

and then premultiplying with  $A^T M$ , whereupon Eqs. (11) are obtained when Eq. (42) is taken into account.

Finally, we consider Eqs. (22), which are equivalent to the matrix equation

$$M \ddot{q} + K q = F \quad (51)$$

where  $F$  is the  $n \times 1$  matrix having  $F_1(t), \dots, F_n(t)$  as elements. Here, expressing  $q$  as in Eq. (17) and then premultiplying with  $A^T$  yields, with the aid of Eqs. (42) and (43),

$$\ddot{Q} + \lambda Q = A^T F \quad (52)$$

which is equivalent to

$$\ddot{Q}_r + \underset{(44, 12) (52)}{p_r^2} Q_r = f_r(t) \quad (r = 1, \dots, n) \quad (53)$$

where  $f_r(t)$  is the  $r$ th element of the  $n \times 1$  matrix  $A^T F$ , that is

$$f_r(t) = \sum_{j=1}^n A_j^{(r)} F_j(t) \quad (r = 1, \dots, n) \quad (54)$$

Now, Eqs. (53) have the same form as Eq. (1) with  $n = 0$ . Referring to Eqs. (2) and (3), we can, therefore, write

$$\begin{aligned} Q_r &= \frac{1}{2ip_r} \left\{ [\dot{Q}_r(0) + ip_r Q_r(0)] e^{ip_r t} - [\dot{Q}_r(0) - ip_r Q_r(0)] e^{-ip_r t} \right. \\ &\quad \left. + \int_0^t f_r(\zeta) [e^{ip_r(t-\zeta)} - e^{-ip_r(t-\zeta)}] d\zeta \right\} \\ &= Q_r(0) \cos p_r t + \frac{\dot{Q}_r(0)}{p_r} \sin p_r t \\ &\quad + \frac{1}{p_r} \int_0^t f_r(\zeta) \sin[p_r(t-\zeta)] d\zeta \quad (r = 1, \dots, n) \end{aligned} \quad (55)$$



or, after using Eqs. (54),

$$\begin{aligned}
 Q_r = & Q_r(0) \cos p_r t + \frac{\dot{Q}_r(0)}{p_r} \sin p_r t \\
 & + \frac{1}{p_r} \sum_{j=1}^n A_j^{(r)} \left[ \sin p_r t \int_0^t F_j(\zeta) \cos p_r \zeta d\zeta \right. \\
 & \left. - \cos p_r t \int_0^t F_j(\zeta) \sin p_r \zeta d\zeta \right] \quad (r = 1, \dots, n)
 \end{aligned} \quad (56)$$

so that, with  $\gamma_j^{(r)}$  and  $\sigma_j^{(r)}$  ( $j, r = 1, \dots, n$ ) as defined in Eqs. (25), and  $\eta_r$  ( $r = 1, \dots, n$ ) as given by Eqs. (24), we have

$$Q_r = Q_r(0) \cos p_r t + \frac{1}{p_r} [\dot{Q}_r(0) \sin p_r t + \eta_r] \quad (r = 1, \dots, n) \quad (57)$$

which is equivalent to Eq. (23) if  $\eta$  is the  $n \times 1$  matrix having  $\eta_1, \dots, \eta_n$  as elements.

**Example** Figure 7.10.1 shows a truss  $T$  consisting of eight members of length  $L$  and mass  $m$  and five members of length  $\sqrt{2}L$  and mass  $\sqrt{2}m$ . All members have the same cross-sectional area  $Z$  and Young's modulus  $E$ , and spherical joints are used to connect members to each other and to a base fixed in a Newtonian reference frame  $N$ . Finally,  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular unit vectors fixed in  $N$ .

When  $T$  is replaced with a *lumped-mass model* for purposes of vibrations analysis, that is, with a set  $S$  of eight particles  $P_1, \dots, P_8$  at the nodes of the

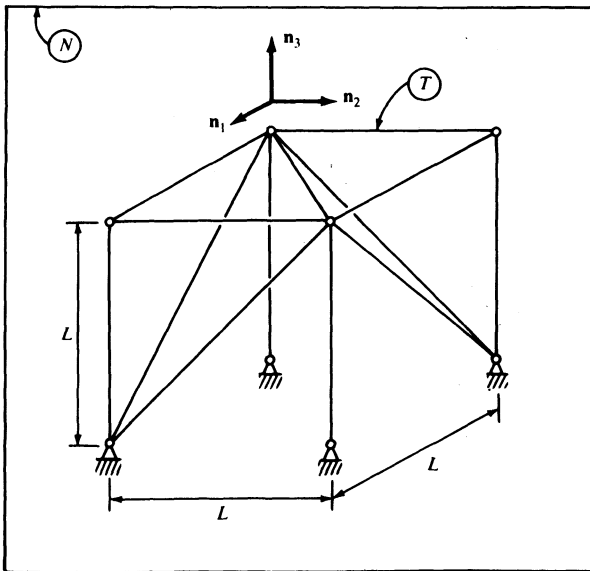


Figure 7.10.1

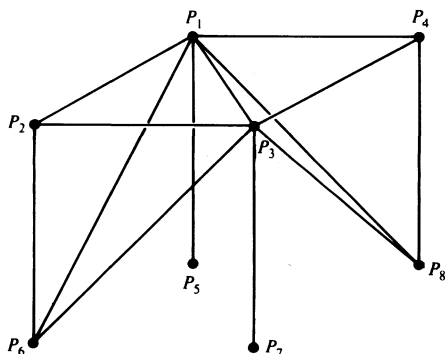


Figure 7.10.2

truss, as indicated in Fig. 7.10.2, then the mass of each particle is taken to be equal to one-half of the sum of the masses of all members meeting at the node where the particle is placed. Thus,  $P_1$  is assigned the mass  $3m(1 + \sqrt{2})/2$ ,  $P_2$  the mass  $3m/2$ , and so forth. As for generalized coordinates, we note that  $S$  possess 12 degrees of freedom in  $N$ ; let  $\delta_i$  be the position vector to  $P_i$  at time  $t$  from the point of  $N$  at which  $P_i$  ( $i = 1, \dots, 4$ ) is situated when  $T$  is undeformed, as shown in Fig. 7.10.3; and define  $q_1, \dots, q_{12}$  as

$$q_r = \left\{ \begin{array}{ll} \delta_1 \cdot \mathbf{n}_r & (r = 1, 2, 3) \\ \delta_2 \cdot \mathbf{n}_{r-3} & (r = 4, 5, 6) \\ \delta_3 \cdot \mathbf{n}_{r-6} & (r = 7, 8, 9) \\ \delta_4 \cdot \mathbf{n}_{r-9} & (r = 10, 11, 12) \end{array} \right\} \quad (58)$$

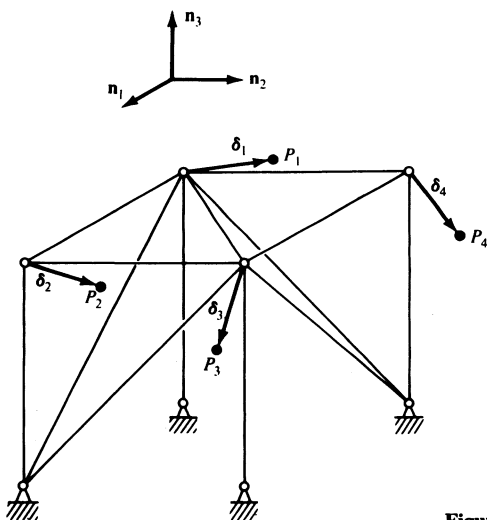


Figure 7.10.3

Linearized in  $q_1, \dots, q_{12}$ , the dynamical equations governing all motions of  $S$  when gravitational forces are treated as negligible are precisely Eqs. (4) or, equivalently, Eqs. (45), with  $n = 12$  and the mass matrix  $M$  given by

$$M = m \begin{bmatrix} m_1 & & & & & & & & & & & \\ & m_1 & & & & & & & & & & \\ & & m_1 & & & & & & & & & \\ & & & m_2 & & & & & & & & \\ & & & & m_2 & & & & & & & \\ & & & & & m_2 & & & & & & \\ & & & & & & m_1 & & & & & \\ & & & & & & & m_1 & & & & \\ & & & & & & & & m_1 & & & \\ & & & & & & & & & m_2 & & \\ & & & & & & & & & & m_2 & \\ & & & & & & & & & & & m_2 \end{bmatrix} \quad (59)$$

where

$$m_1 \triangleq \frac{3(1 + \sqrt{2})}{2} \quad m_2 = \frac{3}{2} \quad (60)$$

while the stiffness matrix  $K$  assumes the form

$$K = k \begin{bmatrix} k_1 & k_2 & -k_2 & -1 & 0 & 0 & -k_2 & -k_2 & 0 & 0 & 0 & 0 \\ k_2 & k_1 & -k_2 & 0 & 0 & 0 & -k_2 & -k_2 & 0 & 0 & -1 & 0 \\ -k_2 & -k_2 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_2 & -k_2 & 0 & 0 & 0 & 0 & k_1 & k_2 & k_2 & -1 & 0 & 0 \\ -k_2 & -k_2 & 0 & 0 & -1 & 0 & k_2 & k_1 & k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_2 & k_2 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (61)$$

with

$$k_1 \triangleq 1 + \frac{\sqrt{2}}{2} \quad k_2 \triangleq \frac{\sqrt{2}}{4} \quad k \triangleq \frac{EZ}{L} \quad (62)$$



where

$$w_1 \triangleq \frac{k_1}{m_1} \quad w_2 \triangleq \frac{k_2}{m_1} \quad w_3 \triangleq \frac{1}{\sqrt{m_1 m_2}} \quad w_4 \triangleq \frac{1}{m_2} \quad (65)$$

The eigenvalues of  $W$  are

$$\lambda_1 = 0.0373322k/m \quad \lambda_2 = 0.0668532k/m \quad \lambda_3 = 0.0668532k/m \quad (66)$$

$$\lambda_4 = 0.244379k/m \quad \lambda_5 = 0.488054k/m \quad \lambda_6 = 0.495117k/m \quad (67)$$

$$\lambda_7 = 0.666667k/m \quad \lambda_8 = 0.666667k/m \quad \lambda_9 = 0.973587k/m \quad (68)$$

$$\lambda_{10} = 0.973587k/m \quad \lambda_{11} = 0.986458k/m \quad \lambda_{12} = 1.16287k/m \quad (69)$$

Note that  $\lambda_2 = \lambda_3$ ,  $\lambda_7 = \lambda_8$ , and  $\lambda_9 = \lambda_{10}$ . Eigenvectors  $C^{(1)}, C^{(2)}, \dots, C^{(12)}$  of  $W$ , corresponding to  $\lambda_1, \lambda_2, \dots, \lambda_{12}$ , respectively, are

$$C^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0.449838 \\ 0.681772 \\ 0.681772 \\ 0 \\ 1 \\ 1 \\ -0.449838 \\ 0.681772 \\ 0.681772 \\ 0 \end{bmatrix}, C^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.715327 \\ 0 \\ -1 \\ 1 \\ 0 \\ -0.715327 \\ 0 \\ 0 \end{bmatrix}, \dots, C^{(12)} = \begin{bmatrix} -1 \\ -1 \\ 0.282388 \\ 0.864687 \\ -0.864687 \\ 0 \\ 1 \\ 1 \\ 0.282388 \\ -0.864687 \\ 0.864687 \\ 0 \end{bmatrix} \quad (70)$$

so that Eqs. (7), (63), (70), and (60) lead to

$$B^{(1)} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.525493 \\ 0.525493 \\ 0.236386 \\ 0.556665 \\ 0.556665 \\ 0 \\ 0.525493 \\ 0.525493 \\ -0.236386 \\ 0.556665 \\ 0.556665 \\ 0 \end{bmatrix}, B^{(2)} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.584062 \\ 0 \\ -0.525493 \\ 0.525493 \\ 0 \\ -0.584062 \\ 0 \\ 0 \end{bmatrix}, \dots, B^{(12)} = \frac{1}{\sqrt{m}} \begin{bmatrix} -0.525493 \\ -0.525493 \\ 0.148393 \\ 0.706014 \\ -0.706014 \\ 0 \\ 0.525493 \\ 0.525493 \\ 0.148393 \\ -0.706014 \\ 0.706014 \\ 0 \end{bmatrix} \quad (71)$$

which means that [see Eqs. (8), (71), and (59)]

$$N_1 = 2.50279, N_2 = 1.73879, \dots, N_{12} = 2.67399 \quad (72)$$

and, from Eqs. (9), (71), and (72),

$$A^{(1)} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.20996 \\ 0.20996 \\ 0.094449 \\ 0.22242 \\ 0.22242 \\ 0 \\ 0.20996 \\ 0.20996 \\ -0.094449 \\ 0.22242 \\ 0.22242 \\ 0 \end{bmatrix}, A^{(2)} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.33590 \\ 0 \\ -0.30222 \\ 0.30222 \\ 0 \\ -0.33590 \\ 0 \\ 0 \end{bmatrix}, \dots, A^{(12)} = \frac{1}{\sqrt{m}} \begin{bmatrix} -0.19652 \\ -0.19652 \\ 0.055495 \\ 0.26403 \\ -0.26403 \\ 0 \\ 0.19652 \\ 0.19652 \\ 0.055495 \\ -0.26403 \\ 0.26403 \\ 0 \end{bmatrix} \quad (73)$$

Referring to Eqs. (10) and (73), one thus finds that the modal matrix  $A$  is given by

$$A = \frac{1}{\sqrt{m}} \begin{bmatrix} \alpha_1 & 0 & -\alpha_5 & \alpha_6 & -\alpha_9 & -\alpha_{12} & 0 & 0 & -\alpha_{16} & 0 & -\alpha_{18} & -\alpha_{21} \\ \alpha_1 & 0 & \alpha_5 & \alpha_6 & -\alpha_9 & -\alpha_{12} & 0 & 0 & \alpha_{16} & 0 & -\alpha_{18} & -\alpha_{21} \\ \alpha_2 & 0 & 0 & \alpha_7 & \alpha_{10} & \alpha_{13} & 0 & 0 & 0 & 0 & \alpha_{19} & \alpha_{22} \\ \alpha_3 & 0 & -\alpha_4 & \alpha_8 & -\alpha_{11} & -\alpha_{14} & 0 & 0 & \alpha_{17} & 0 & \alpha_{20} & \alpha_{23} \\ \alpha_3 & \alpha_4 & 0 & -\alpha_8 & -\alpha_{11} & \alpha_{14} & 0 & 0 & 0 & -\alpha_{17} & \alpha_{20} & -\alpha_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{15} & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & -\alpha_5 & 0 & -\alpha_6 & -\alpha_9 & \alpha_{12} & 0 & 0 & 0 & -\alpha_{16} & -\alpha_{18} & \alpha_{21} \\ \alpha_1 & \alpha_5 & 0 & -\alpha_6 & -\alpha_9 & \alpha_{12} & 0 & 0 & 0 & \alpha_{16} & -\alpha_{18} & \alpha_{21} \\ -\alpha_2 & 0 & 0 & \alpha_7 & -\alpha_{10} & \alpha_{13} & 0 & 0 & 0 & 0 & -\alpha_{19} & \alpha_{22} \\ \alpha_3 & -\alpha_4 & 0 & -\alpha_8 & -\alpha_{11} & \alpha_{14} & 0 & 0 & 0 & \alpha_{17} & \alpha_{20} & -\alpha_{23} \\ \alpha_3 & 0 & \alpha_4 & \alpha_8 & -\alpha_{11} & -\alpha_{14} & 0 & 0 & -\alpha_{17} & 0 & \alpha_{20} & \alpha_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{15} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (74)$$

where  $\alpha_1, \dots, \alpha_{23}$  have the values

$$\alpha_1 = 0.20996 \quad \alpha_2 = 0.094449 \quad \alpha_3 = 0.22242 \quad \alpha_4 = 0.33590 \quad (75)$$

$$\alpha_5 = 0.30222 \quad \alpha_6 = 0.16952 \quad \alpha_7 = 0.14581 \quad \alpha_8 = 0.26763 \quad (76)$$

$$\alpha_9 = 0.030231 \quad \alpha_{10} = 0.35454 \quad \alpha_{11} = 0.11284 \quad \alpha_{12} = 0.040955 \quad (77)$$

$$\alpha_{13} = 0.33724 \quad \alpha_{14} = 0.15916 \quad \alpha_{15} = 0.81650 \quad \alpha_{16} = 0.21618 \quad (78)$$

$$\alpha_{17} = 0.46958 \quad \alpha_{18} = 0.15504 \quad \alpha_{19} = 0.058777 \quad \alpha_{20} = 0.32321 \quad (79)$$

$$\alpha_{21} = 0.19652 \quad \alpha_{22} = 0.055495 \quad \alpha_{23} = 0.26403 \quad (80)$$

Suppose that, at  $t = 0$ ,  $\delta_1 = \delta_2 = \delta_3 = \delta_4 = (L/10)\mathbf{n}_1$ , and  $P_1, \dots, P_4$  are at rest. Then, in accordance with Eqs. (58),

$$q(0) = \left(\frac{L}{10}\right)[1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0]^T \quad (81)$$

and

$$\dot{q}(0) = 0 \quad (82)$$

Consequently,

$$Q(0) \underset{(11,74,59,81)}{=} \sqrt{mL} \begin{bmatrix} 0.2188 & -0.1598 & -0.1598 & 0 & -0.05575 & 0 & 0 \\ 0 & -0.007851 & -0.007851 & -0.01533 & 0 \end{bmatrix} \quad (83)$$

$$\dot{Q}(0) \underset{(11,82)}{=} 0 \quad (84)$$

and, after noting that [see Eqs. (12) and (66)–(69)]

$$p_1 = 0.1932\omega \quad p_2 = 0.2586\omega \quad p_3 = 0.2586\omega \quad (85)$$

$$p_4 = 0.4943\omega \quad p_5 = 0.6986\omega \quad p_6 = 0.7036\omega \quad (86)$$

$$p_7 = 0.8165\omega \quad p_8 = 0.8165\omega \quad p_9 = 0.9867\omega \quad (87)$$

$$p_{10} = 0.9867\omega \quad p_{11} = 0.9932\omega \quad p_{12} = 1.078\omega \quad (88)$$

where  $\omega$  is defined as

$$\omega \triangleq \sqrt{\frac{k}{m}} \quad (89)$$

one has

$$Q_{(16,13,83-88)} = \sqrt{mL} \begin{bmatrix} 0.2188 \cos(0.1932\omega t) \\ -0.1598 \cos(0.2586\omega t) \\ -0.1598 \cos(0.2586\omega t) \\ 0 \\ -0.05575 \cos(0.6986\omega t) \\ 0 \\ 0 \\ 0 \\ -0.007851 \cos(0.9867\omega t) \\ -0.007851 \cos(0.9867\omega t) \\ -0.01533 \cos(0.9932\omega t) \\ 0 \end{bmatrix} \quad (90)$$

and Eqs. (17), (73), and (90) thus lead, for example, to

$$\frac{q_1}{L} = \sum_{i=1}^{12} x_1^{(i)} \quad (91)$$

where

$$x_1^{(1)} = 0.04594 \cos(0.1932\omega t) \quad x_1^{(2)} = 0 \quad (92)$$

$$x_1^{(3)} = 0.04830 \cos(0.2586\omega t) \quad x_1^{(4)} = 0 \quad (93)$$

$$x_1^{(5)} = 0.001685 \cos(0.6986\omega t) \quad x_1^{(6)} = x_1^{(7)} = x_1^{(8)} = 0 \quad (94)$$

$$x_1^{(9)} = 0.001697 \cos(0.9867\omega t) \quad x_1^{(10)} = 0 \quad (95)$$

$$x_1^{(11)} = 0.002376 \cos(0.9932\omega t) \quad x_1^{(12)} = 0 \quad (96)$$

and

$$\frac{q_{10}}{L} = \sum_{i=1}^{12} x_{10}^{(i)} \quad (97)$$

where

$$x_{10}^{(1)} = 0.04866 \cos(0.1932\omega t) \quad x_{10}^{(2)} = 0.05369 \cos(0.258\omega t) \quad (98)$$

$$x_{10}^{(3)} = x_{10}^{(4)} = 0 \quad x_{10}^{(5)} = 0.006290 \cos(0.6986\omega t) \quad (99)$$

$$x_{10}^{(6)} = x_{10}^{(7)} = 0 \quad (100)$$

$$x_{10}^{(8)} = x_{10}^{(9)} = 0 \quad x_{10}^{(10)} = -0.003686 \cos(0.9867\omega t) \quad (101)$$

$$x_{10}^{(11)} = -0.004954 \cos(0.9932\omega t) \quad x_{10}^{(12)} = 0 \quad (102)$$



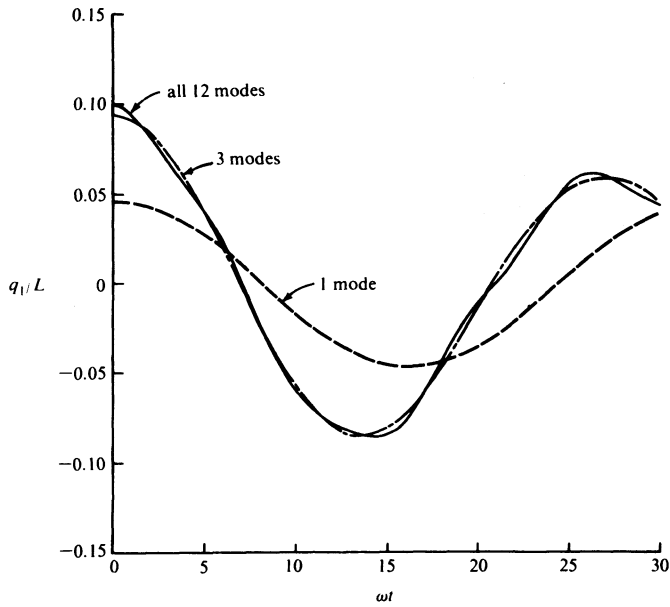


Figure 7.10.4

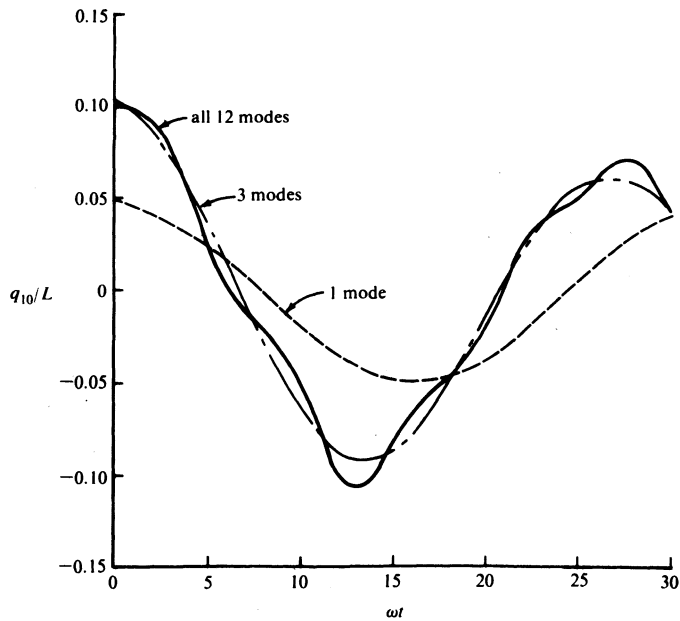


Figure 7.10.5

Figure 7.10.4 contains three plots of  $q_1/L$  versus  $\omega t$ . The curve labeled "all 12 modes" is based on Eq. (91), and the curves labeled "3 modes" and "1 mode" correspond, respectively, to using only the first three terms and the first term of Eq. (91); similarly for Fig. 7.10.5,  $q_{10}/L$ , and Eq. (97). As can be seen, use of only the first mode leads to rather poor approximations, whereas results obtained by truncation after the first three modes may be acceptable.

## PROBLEM SETS

### PROBLEM SET 1

(Secs. 1.1–1.9)

**1.1** Four rectangular parallelepipeds,  $A$ ,  $B$ ,  $C$ , and  $D$ , are arranged as shown in Fig. P1.1, where  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  are unit vectors parallel to edges of  $A$ ,  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$  are unit vectors parallel to edges of  $B$ , and so forth, while  $q_1$ ,  $q_2$ , and  $q_3$  are the radian measures of angles that determine the relative orientations of  $A$ ,  $\dots$ ,  $D$ . The configuration shown is one for which  $q_1$ ,  $q_2$ , and  $q_3$  are positive.

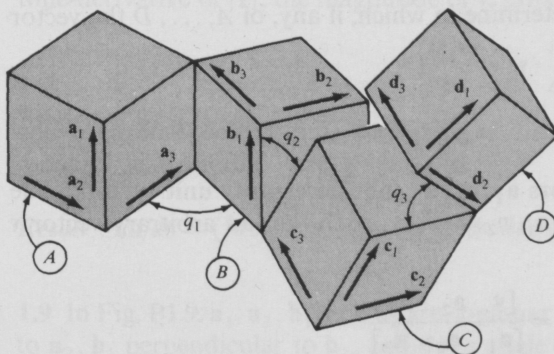


Figure P1.1

Vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are given by

$$\mathbf{u} = \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3$$

$$\mathbf{v} = \mathbf{b}_1 + \mathbf{c}_2 + \mathbf{d}_3$$

$$\mathbf{w} = \mathbf{d}_1 + 2\mathbf{d}_2 + q_3\mathbf{d}_3$$

For each of the vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ , for each of the variables  $q_1$ ,  $q_2$ , and  $q_3$ , and for each of the reference frames  $A, \dots, D$ , determine whether or not the vector is a function of the variable in the reference frame. Use the letters Y and N, standing for “yes” and “no,” respectively, to indicate your results, as shown below.

### Results

$A$	$q_1$	$q_2$	$q_3$
$\mathbf{u}$	N	N	N
$\mathbf{v}$	Y	Y	N
$\mathbf{w}$	Y	Y	Y

$B$	$q_1$	$q_2$	$q_3$
$\mathbf{u}$	Y	N	N
$\mathbf{v}$	N	Y	N
$\mathbf{w}$	N	Y	Y

$C$	$q_1$	$q_2$	$q_3$
$\mathbf{u}$	Y	Y	N
$\mathbf{v}$	N	Y	N
$\mathbf{w}$	N	N	Y

$D$	$q_1$	$q_2$	$q_3$
$\mathbf{u}$	Y	Y	Y
$\mathbf{v}$	N	Y	Y
$\mathbf{w}$	N	N	Y

**1.2** Referring to Problem 1.1, and supposing that  $q_1$  is a function of time  $t$ , whereas  $q_2$  and  $q_3$  are independent of  $t$ , determine in which, if any, of  $A, \dots, D$  the vector  $\mathbf{v}$  is a function of  $t$ .

### Result A

**1.3** Referring to Problem 1.1, and supposing that  $q_2$  is a function of time  $t$ , whereas  $q_1$  and  $q_3$  are independent of  $t$ , determine in which, if any, of  $A, \dots, D$  the vector  $\mathbf{v}$  is a function of  $t$ .

### Result A, B, C, D

**1.4** Given any noncoplanar vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  (not necessarily unit vectors), one can find (uniquely) three scalars,  $v_1, v_2$ , and  $v_3$ , such that an arbitrary vector  $\mathbf{v}$  is given by Eq. (1.3.1). Show that

$$v_1 = \frac{[\mathbf{v} \ \mathbf{a}_2 \ \mathbf{a}_3]}{[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]}$$

where, for any vectors  $\alpha$ ,  $\beta$ , and  $\gamma$ , the quantity  $[\alpha \ \beta \ \gamma]$ , called the *scalar triple product* of  $\alpha$ ,  $\beta$ , and  $\gamma$ , is defined as

$$[\alpha \ \beta \ \gamma] \triangleq \alpha \cdot \beta \times \gamma$$

Verify that  $\alpha \times \beta \cdot \gamma = [\alpha \ \beta \ \gamma]$  and that  $[\alpha \ \beta \ \gamma] = [\gamma \ \alpha \ \beta] = [\beta \ \gamma \ \alpha]$ .

**1.5** Referring to Problem 1.1, determine the magnitude of each of the following partial derivatives:  ${}^A\partial\mathbf{v}/\partial q_1$ ,  ${}^B\partial\mathbf{v}/\partial q_1$ ,  ${}^C\partial\mathbf{v}/\partial q_2$ ,  ${}^C\partial\mathbf{v}/\partial q_3$ ,  ${}^D\partial\mathbf{v}/\partial q_2$ ,  ${}^D\partial\mathbf{v}/\partial q_1$ .

**Results**  $(1 + \cos^2 q_2)^{1/2}$ , 0, 1, 0, 1, 0

**1.6** Referring to Problem 1.1, determine (a) the  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  measure numbers of  ${}^B\partial\mathbf{u}/\partial q_1$ , (b) the  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$  measure numbers of  ${}^B\partial\mathbf{u}/\partial q_1$ , and (c) the  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  measure numbers of  ${}^A\partial\mathbf{u}/\partial q_1$ .

**Results** (a) 0, 3, -2 (b) 0,  $-2 \sin q_1 + 3 \cos q_1$ ,  $-2 \cos q_1 - 3 \sin q_1$  (c) 0, 0, 0

**1.7** The position vector  $\mathbf{r}^{PQ}$  from a point  $P$  fixed in  $A$  to a point  $Q$  fixed in  $D$ , where  $A$  and  $D$  are two of the parallelepipeds introduced in Problem 1.1 and shown in Fig. P1.1, is given by

$$\mathbf{r}^{PQ} = \alpha \mathbf{a}_1 + \beta \mathbf{b}_2 + \gamma \mathbf{c}_3$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the following functions of  $q_1$ ,  $q_2$ , and  $q_3$ :

$$\alpha = q_1 + q_2 + q_3 \quad \beta = q_1^2 + q_2^2 + q_3^2 \quad \gamma = q_1^3 + q_2^3 + q_3^3$$

Determine the magnitude of  ${}^D\partial\mathbf{r}^{PQ}/\partial q_3$  for  $q_1 = \pi/2$  rad,  $q_2 = q_3 = 0$ .

**Result**  $[1 + 3(\pi/2)^2 + (\pi/2)^4]^{1/2}$

**\*1.8** A vector  $\mathbf{v}$  is a function of time  $t$  in a reference frame  $A$ . Show that the first time-derivative of  $|\mathbf{v}|$ , the magnitude of  $\mathbf{v}$ , satisfies the equation

$$|\mathbf{v}| \frac{d|\mathbf{v}|}{dt} = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

and determine the first time-derivative in  $A$  of a unit vector  $\mathbf{u}$  that has the same direction as  $\mathbf{v}$  for all  $t$ .

**Result**  ${}^A d\mathbf{u}/dt = ({}^A d\mathbf{v}/dt)|\mathbf{v}|^{-1} - \mathbf{v}\mathbf{v} \cdot ({}^A d\mathbf{v}/dt)|\mathbf{v}|^{-3}$

**1.9** In Fig. P1.9,  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{b}_1$ , and  $\mathbf{b}_2$  are coplanar unit vectors, with  $\mathbf{a}_1$  perpendicular to  $\mathbf{a}_2$ ,  $\mathbf{b}_1$  perpendicular to  $\mathbf{b}_2$ , and  $\theta$  the angle between  $\mathbf{a}_1$  and  $\mathbf{b}_1$ . Letting  $A$  be a

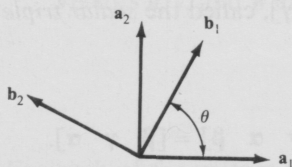


Figure P1.9

reference frame in which  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are fixed,  $B$  a reference frame in which  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are fixed, and  $\mathbf{v}$  a vector given by

$$\mathbf{v} = f\mathbf{a}_1 + g\mathbf{a}_2$$

where  $f$  and  $g$  are functions of two scalar variables  $q_1$  and  $q_2$ , show that, if  $\theta$  is also a function of  $q_1$  and  $q_2$ , then

$$\frac{{}^A\partial}{{\partial}q_1} \left( \frac{{}^B\partial\mathbf{v}}{{\partial}q_2} \right) - \frac{{}^B\partial}{{\partial}q_2} \left( \frac{{}^A\partial\mathbf{v}}{{\partial}q_1} \right) = (g\mathbf{a}_1 - f\mathbf{a}_2) \frac{\partial^2\theta}{{\partial}q_1{\partial}q_2}$$

**1.10** A circular disk  $C$  of radius  $r$  can rotate about an axis  $X$  fixed in a laboratory  $L$ , as shown in Fig. P1.10, and a rod  $R$  of length  $3r$  is pinned to  $C$ , the axis  $Y$  of the pin passing through the center  $O$  of  $C$ . Letting  $\mathbf{p}$  be the position vector from  $O$  to  $P$ , the endpoint of  $R$ , express the first time-derivative of  $\mathbf{p}$  in  $L$  in terms of  $q_1, q_2, \dot{q}_1, \dot{q}_2, \mathbf{c}_1, \mathbf{c}_2$ , and  $\mathbf{c}_3$ , where  $q_1$  and  $q_2$  are the radian measures of angles, as indicated in Fig. P1.9,  $\dot{q}_1$  and  $\dot{q}_2$  denote the time-derivatives of  $q_1$  and  $q_2$ , and  $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$  are unit vectors fixed in  $C$  and directed as shown.

*Suggestion:* First verify that

$$\frac{{}^L\partial\mathbf{c}_2}{{\partial}q_1} = \mathbf{c}_3 \quad \frac{{}^L\partial\mathbf{c}_3}{{\partial}q_1} = -\mathbf{c}_2$$

**Result**  $r[\dot{q}_1(\mathbf{c}_3 - 3 \sin q_2 \mathbf{c}_2) + 3\dot{q}_2(\cos q_2 \mathbf{c}_3 + \sin q_2 \mathbf{c}_1)]$

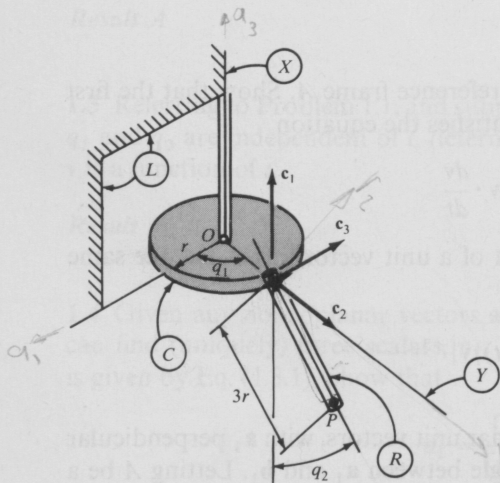


Figure P1.10

**\*1.11** A vector  $\mathbf{v}$  is a function of time  $t$  in a reference frame  $N$  in which mutually perpendicular unit vectors  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are fixed. At time  $t^*$ ,

$$\mathbf{v} = \mathbf{n}_1 \quad \frac{{}^N d\mathbf{v}}{dt} = \mathbf{n}_2 \quad \frac{{}^N d^2\mathbf{v}}{dt^2} = \mathbf{n}_3$$

Letting  $\mathbf{u}$  be a unit vector that has the same direction as  $\mathbf{v}$  at all times, determine the magnitude of  ${}^N d^2\mathbf{u}/dt^2$  at time  $t^*$ .

**Result**  $\sqrt{2}$

## PROBLEM SET 2

(Secs. 2.1–2.5)

**2.1** When a point  $P$  moves on a space curve  $C$  fixed in a reference frame  $A$ , a dextral set of orthogonal unit vectors  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  can be generated by letting  $\mathbf{p}$  be the position vector from a point  $O$  fixed on  $C$  to the point  $P$  and defining  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_3$  as

$$\mathbf{b}_1 \triangleq \mathbf{p}' \quad \mathbf{b}_2 \triangleq \frac{\mathbf{p}''}{|\mathbf{p}''|} \quad \mathbf{b}_3 \triangleq \mathbf{b}_1 \times \mathbf{b}_2$$

where primes denote differentiation in  $A$  with respect to the length  $s$  of the arc of  $C$  that connects  $O$  to  $P$  (see Fig. P2.1). The vector  $\mathbf{b}_1$  is called a *vector tangent*,  $\mathbf{b}_2$  the *vector principal normal*, and  $\mathbf{b}_3$  a *vector binormal* of  $C$  at  $P$ , and the derivatives in  $A$  of  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_3$  with respect to  $s$  are given by the Serret–Frénet formulas

$$\mathbf{b}_1' = \frac{\mathbf{b}_2}{\rho} \quad \mathbf{b}_2' = -\frac{\mathbf{b}_1}{\rho} + \lambda \mathbf{b}_3 \quad \mathbf{b}_3' = -\lambda \mathbf{b}_2$$

where  $\rho$  and  $\lambda$ , defined as

$$\rho \triangleq \frac{1}{|\mathbf{p}''|} \quad \lambda \triangleq \rho^2 \mathbf{p}' \cdot \mathbf{p}'' \times \mathbf{p}'''$$

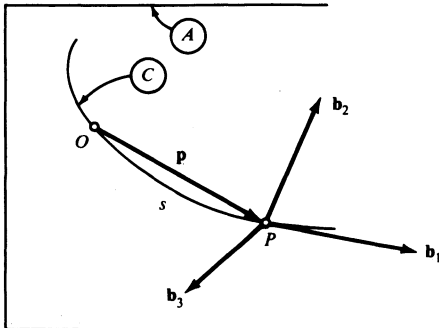


Figure P2.1

are called the *principal radius of curvature* of  $C$  at  $P$  and the *torsion* of  $C$  at  $P$ , respectively.

Letting  $B$  be a rigid body in which  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are fixed, express the angular velocity of  $B$  in  $A$  in terms of  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \rho, \lambda$ , and  $\dot{s}$ , the time-derivative of  $s$ , taking advantage of the fact that

$$\frac{{}^A d\mathbf{b}_i}{dt} = \mathbf{b}_i' \dot{s} \quad (i = 1, 2, 3)$$

**Result**  $(\lambda \mathbf{b}_1 + \mathbf{b}_3/\rho)\dot{s}$

**2.2** Table P2.2 shows the relationship between two dextral sets of orthogonal unit vectors,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  and  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ , fixed in rigid bodies  $A$  and  $B$ , respectively. The symbols  $s_i$  and  $c_i$  in the table stand for  $\sin q_i$  and  $\cos q_i$ , respectively, where  $q_i$  ( $i = 1, 2, 3$ ) are the radian measures of certain angles.

**Table P2.2**

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$
$\mathbf{a}_1$	$c_2 c_3$	$s_1 s_2 c_3 - s_3 c_1$	$c_1 s_2 c_3 + s_3 s_1$
$\mathbf{a}_2$	$c_2 s_3$	$s_1 s_2 s_3 + c_3 c_1$	$c_1 s_2 s_3 - c_3 s_1$
$\mathbf{a}_3$	$-s_2$	$s_1 c_2$	$c_1 c_2$

Determine  $\alpha_2$  and  $\beta_2$  such that  $\boldsymbol{\omega}$ , the angular velocity of  $B$  in  $A$ , is given by

$$\boldsymbol{\omega} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3$$

**Results**  $\dot{q}_1 c_2 s_3 + \dot{q}_2 c_3 \quad \dot{q}_2 c_1 + \dot{q}_3 s_1 c_2$

**2.3** Referring to Problem 1.1, use the definition of angular velocity given in Sec. 2.1 to show that the angular velocity of  $B$  in  $D$  can be expressed as

$${}^D \boldsymbol{\omega}^B = -(\dot{q}_3 \sin q_2 \mathbf{b}_1 + \dot{q}_2 \mathbf{b}_2 + \dot{q}_3 \cos q_2 \mathbf{b}_3)$$

Next, form  ${}^D \boldsymbol{\omega}^C$  and  ${}^C \boldsymbol{\omega}^B$ , and then verify that  ${}^D \boldsymbol{\omega}^B = {}^D \boldsymbol{\omega}^C + {}^C \boldsymbol{\omega}^B$ , in agreement with the addition theorem for angular velocities. Finally, using the notations  $s_i = \sin q_i$ ,  $c_i = \cos q_i$  ( $i = 1, 2, 3$ ), determine  $\omega_i$ , defined as  $\omega_i \triangleq {}^A \boldsymbol{\omega}^D \cdot \mathbf{d}_i$  ( $i = 1, 2, 3$ ).

**Results**  $\dot{q}_1 c_2 c_3 + \dot{q}_2 s_3 \quad -\dot{q}_1 c_2 s_3 + \dot{q}_2 c_3 \quad \dot{q}_1 s_2 + \dot{q}_3$



2.4 Letting  $\omega$  be the angular velocity of a rigid body  $B$  in a reference frame  $A$ , show that the angular velocity of  $A$  in  $B$  is equal to  $-\omega$ , and that

$$\frac{{}^A d\omega}{{}^A dt} = \frac{{}^B d\omega}{{}^B dt}$$

\*2.5 Letting  $\beta_1$  and  $\beta_2$  be nonparallel vectors fixed in a rigid body  $B$ , and using dots to denote differentiation with respect to time in a reference frame  $A$ , show that the angular velocity  $\omega$  of  $B$  in  $A$  can be expressed as

$$\omega = \frac{\dot{\beta}_1 \times \dot{\beta}_2}{\beta_1 \cdot \beta_2} = \frac{1}{2} \left( \frac{\dot{\beta}_1 \times \dot{\beta}_2}{\beta_1 \cdot \beta_2} + \frac{\dot{\beta}_2 \times \dot{\beta}_1}{\beta_2 \cdot \beta_1} \right)$$

*Suggestion:* Make use of the fact that, for any vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ ,  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{c} \mathbf{b} - \mathbf{a} \cdot \mathbf{b} \mathbf{c}$ .

2.6 Referring to Problem 2.1, suppose that  $\mathbf{q}$ , the position vector from  $P$  to a point  $Q$  moving in  $A$ , is given by

$$\mathbf{q} = q_1 \mathbf{b}_1 + q_2 \mathbf{b}_2 + q_3 \mathbf{b}_3$$

where  $q_1, q_2, q_3$  are functions of time  $t$ . Determine the quantities  $v_i$  ( $i = 1, 2, 3$ ) such that the first time-derivative in  $A$  of the position vector  $\mathbf{r}$  from  $O$  to  $Q$  is equal to  $v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + v_3 \mathbf{b}_3$ .

**Results**  $\dot{q}_1 + \dot{s}(1 - q_2/\rho)$   $\dot{q}_2 + \dot{s}[(q_1/\rho) - \lambda q_3]$   $\dot{q}_3 + \dot{s}\lambda q_2$

2.7 Figure P2.7 shows a circular disk  $C$  of radius  $R$  in contact with a horizontal plane  $H$  that is fixed in a reference frame  $A$  rigidly attached to the Earth. Mutually

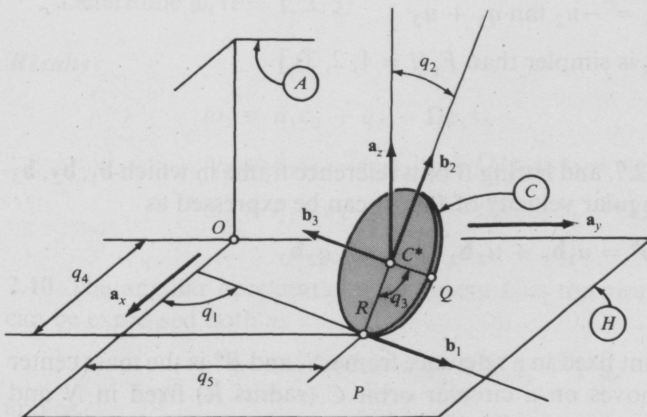


Figure P2.7

perpendicular unit vectors  $\mathbf{a}_x$ ,  $\mathbf{a}_y$ , and  $\mathbf{a}_z \triangleq \mathbf{a}_x \times \mathbf{a}_y$  are fixed in  $A$ , and  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$  form a dextral set of orthogonal unit vectors, with  $\mathbf{b}_1$  parallel to the tangent to the periphery of  $C$  at the point of contact between  $A$  and  $C$ ,  $\mathbf{b}_2$  parallel to the line connecting this contact point to  $C^*$ , the center of  $C$ , and  $\mathbf{b}_3$  normal to the plane of  $C$ .

The orientation of  $C$  in  $A$  can be described in terms of the three angles  $q_1$ ,  $q_2$ ,  $q_3$  indicated in Fig. P2.7, where  $Q$  is a point fixed on the periphery of  $C$ . The two quantities  $q_4$  and  $q_5$  characterize the position in  $A$  of the *path point*  $P$ .

The angular velocity of  $C$  in  $A$  can be expressed both as

$${}^A\boldsymbol{\omega}^C = u_x \mathbf{a}_x + u_y \mathbf{a}_y + u_z \mathbf{a}_z$$

and as

$${}^A\boldsymbol{\omega}^C = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_3 \mathbf{b}_3$$

where  $u_x$ ,  $u_y$ ,  $u_z$  and  $u_1$ ,  $u_2$ ,  $u_3$  are functions of  $q_i$  and  $\dot{q}_i$  ( $i = 1, 2, 3$ ). Concomitantly,  $\dot{q}_i$  ( $i = 1, 2, 3$ ) can be expressed as a function  $F_i$  of  $q_1$ ,  $q_2$ ,  $q_3$ ,  $u_x$ ,  $u_y$ ,  $u_z$  or as a function  $G_i$  of  $q_1$ ,  $q_2$ ,  $q_3$ ,  $u_1$ ,  $u_2$ ,  $u_3$ . Determine  $F_i$  and  $G_i$  ( $i = 1, 2, 3$ ). [The equations  $\dot{q}_i = F_i$  and  $\dot{q}_i = G_i$  ( $i = 1, 2, 3$ ) are called *kinematical differential equations*. When  $u_x$ ,  $u_y$ ,  $u_z$  or  $u_1$ ,  $u_2$ ,  $u_3$  are known as functions of  $t$ , these equations can be solved for  $q_1$ ,  $q_2$ ,  $q_3$  to obtain a description of the orientation of  $C$  in  $A$ .]

### Results

$$F_1 = (-u_x s_1 + u_y c_1) \tan q_2 + u_z$$

$$F_2 = -u_x c_1 - u_y s_1$$

$$F_3 = (u_x s_1 - u_y c_1) \sec q_2$$

$$G_1 = u_2 \sec q_2$$

$$G_2 = -u_1$$

$$G_3 = -u_2 \tan q_2 + u_3$$

[It is worth noting that  $G_i$  is simpler than  $F_i$  ( $i = 1, 2, 3$ ).]

**2.8** Referring to Problem 2.7, and letting  $B$  be a reference frame in which  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$  are fixed, show that the angular velocity of  $B$  in  $A$  can be expressed as

$${}^A\boldsymbol{\omega}^B = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_2 \tan q_2 \mathbf{b}_3$$

**2.9** In Fig. P2.9,  $O$  is a point fixed in a reference frame  $N$ , and  $B^*$  is the mass center of a rigid body  $B$  that moves on a circular orbit  $C$  (radius  $R$ ) fixed in  $N$  and centered at  $O$ .  $A_1$ ,  $A_2$ ,  $A_3$  are mutually perpendicular lines,  $A_1$  passing through  $O$  and  $B^*$ ,  $A_2$  tangent to  $C$  at  $B^*$ , and  $A_3$  thus being normal to the plane of  $C$ .

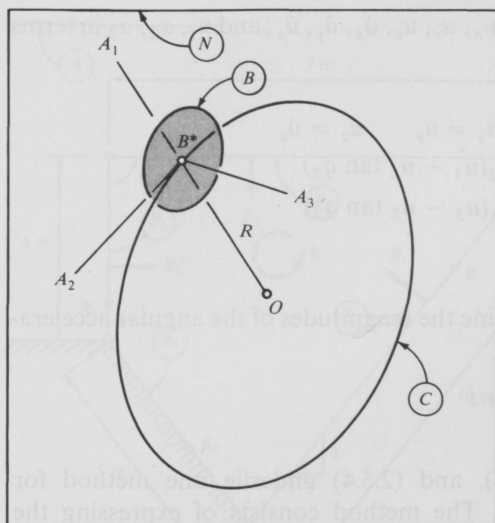


Figure P2.9

If  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a dextral set of mutually perpendicular unit vectors fixed in  $B$ , the “attitude” of  $B$  in a reference frame  $A$  in which  $A_1, A_2, A_3$  are fixed can be specified in terms of  $q_1, q_2, q_3$ , the radian measures of three angles generated as follows: Let  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  be a dextral set of mutually perpendicular unit vectors fixed in  $A$  with  $\mathbf{a}_i$  parallel to  $A_i$  ( $i = 1, 2, 3$ ), align  $\mathbf{b}_i$  with  $\mathbf{a}_i$  ( $i = 1, 2, 3$ ), and then subject  $B$  to successive right-handed rotations relative to  $A$ , characterized by  $q_1\mathbf{b}_1, q_2\mathbf{b}_2$ , and  $q_3\mathbf{b}_1$  (note the last subscript). The quantities  $\omega_i$  ( $i = 1, 2, 3$ ), defined as

$$\omega_i \triangleq \boldsymbol{\omega} \cdot \mathbf{b}_i \quad (i = 1, 2, 3)$$

where  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $N$ , then can be expressed as functions of  $q_i, \dot{q}_i$  ( $i = 1, 2, 3$ ), and  $\Omega$ , if the angular velocity of  $A$  in  $N$  is given by  $\Omega\mathbf{a}_3$ .

Determine  $\omega_i$  ( $i = 1, 2, 3$ ).

### Results

$$\omega_1 = \dot{q}_1 c_2 + \dot{q}_3 - \Omega c_1 s_2$$

$$\omega_2 = \dot{q}_1 s_2 s_3 + \dot{q}_2 c_3 + \Omega (c_1 c_2 s_3 + c_3 s_1)$$

$$\omega_3 = \dot{q}_1 s_2 c_3 - \dot{q}_2 s_3 + \Omega (c_1 c_2 c_3 - s_3 s_1)$$

**2.10** The angular acceleration  ${}^A\boldsymbol{\alpha}^C$ , where  $C$  is the circular disk in Problem 2.7, can be expressed both as

$${}^A\boldsymbol{\alpha}^C = \alpha_x \mathbf{a}_x + \alpha_y \mathbf{a}_y + \alpha_z \mathbf{a}_z$$

and as

$${}^A\boldsymbol{\alpha}^C = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \alpha_3 \mathbf{b}_3$$

Express  $\alpha_x, \alpha_y, \alpha_z$  in terms of  $q_1, q_2, q_3, u_x, u_y, u_z, \dot{u}_x, \dot{u}_y, \dot{u}_z$ ; and  $\alpha_1, \alpha_2, \alpha_3$  in terms of  $q_i, u_i, \dot{u}_i$  ( $i = 1, 2, 3$ ).

**Results**

$$\begin{aligned}\alpha_x &= \dot{u}_x & \alpha_y &= \dot{u}_y & \alpha_z &= \dot{u}_z \\ \alpha_1 &= \dot{u}_1 + u_2(u_3 - u_2 \tan q_2) \\ \alpha_2 &= \dot{u}_2 - u_1(u_3 - u_2 \tan q_2) \\ \alpha_3 &= \dot{u}_3\end{aligned}$$

**2.11** Referring to Problem 1.10, determine the magnitudes of the angular acceleration of  $C$  in  $L$ ,  $R$  in  $C$ , and  $R$  in  $L$ .

**Results**  $|\ddot{q}_1|, |\ddot{q}_2|, [\ddot{q}_1^2 + \ddot{q}_2^2 + (\dot{q}_1 \dot{q}_2)^2]^{1/2}$

**\*2.12** Equations (2.1.2), (2.2.1), (2.5.3), and (2.5.4) underlie one method for analyzing motions of planar linkages. The method consists of expressing the position vector from a hinge point of the linkage to the same hinge point as the sum of position vectors from one hinge point to an adjacent one, setting this sum equal to zero, differentiating the resulting equation repeatedly with respect to time  $t$ , and making use of the relationships

$$\begin{aligned}{}^A\boldsymbol{\omega}^{B_i} &= \omega_i \mathbf{k} & {}^A\boldsymbol{\alpha}^{B_i} &= \alpha_i \mathbf{k}\end{aligned}\begin{array}{l} (2.2.1) \\ (2.5.3) \end{array}$$

$$\frac{{}^A d\boldsymbol{\beta}_i}{dt} = \omega_i \mathbf{k} \times \boldsymbol{\beta}_i = \omega_i \boldsymbol{\beta}_i' \quad (2.1.2)$$

$$\frac{{}^A d\boldsymbol{\beta}_i'}{dt} = \omega_i \mathbf{k} \times \boldsymbol{\beta}_i' = -\omega_i \boldsymbol{\beta}_i \quad (2.1.2)$$

and

$$\alpha_i = \frac{d\omega_i}{dt} \quad (2.5.4)$$

where  $A$  is a reference frame in which the plane of the linkage is fixed,  $B_i$  is a typical member of the linkage,  $\mathbf{k}$  is a unit vector normal to the plane of the linkage,  $\boldsymbol{\beta}_i$  is a unit vector parallel to the line connecting the hinge points of  $B_i$ ,  $\boldsymbol{\beta}_i'$  is the unit vector  $\mathbf{k} \times \boldsymbol{\beta}_i$ , and  $\omega_i$  and  $\alpha_i$  are an angular speed of  $B_i$  (see Sec. 2.2) and a scalar angular acceleration of  $B_i$  (see Sec. 2.5), respectively.

Figure P2.12 shows the configuration at time  $t^*$  of a four-bar linkage, one bar of which,  $B_4$ , is fixed in a reference frame  $A$ . At time  $t^*$ , the angular velocity of  $B_3$  in  $A$  and the angular acceleration of  $B_1$  in  $A$  are equal to  $-6\mathbf{k}$  rad/s and  $5\mathbf{k}$  rad/s<sup>2</sup>, respectively. Determine the angular velocities of  $B_1$  and  $B_2$  in  $A$ , and the angular accelerations of  $B_2$  and  $B_3$  in  $A$ , at time  $t^*$ , by making use of the equation

$$10\boldsymbol{\beta}_1 + 9\boldsymbol{\beta}_2 + 4\boldsymbol{\beta}_3 + 5\boldsymbol{\beta}_4 = 0$$

which is valid for all  $t$ .

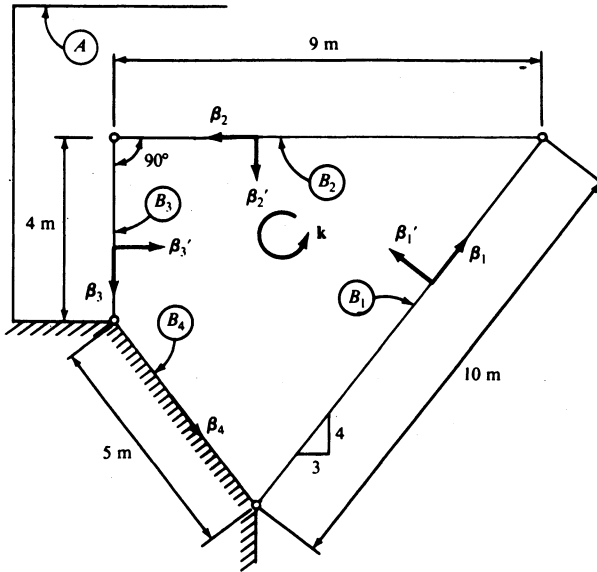


Figure P2.12

### Results

$$\begin{aligned} {}^A\omega^{B_1} &= -3\mathbf{k} \text{ rad/s} & {}^A\omega^{B_2} &= -2\mathbf{k} \text{ rad/s} \\ {}^A\alpha^{B_2} &= (34/3)\mathbf{k} \text{ rad/s}^2 & {}^A\alpha^{B_3} &= (29/2)\mathbf{k} \text{ rad/s}^2 \end{aligned}$$

## PROBLEM SET 3

(Secs. 2.6–2.8)

3.1 Apply the definitions given in Eqs. (2.6.1) and (2.6.2) to formulate expressions for the velocity and the acceleration of the endpoint  $P$  of the rod  $R$  of Problem 1.10 in (a) reference frame  $L$  and (b) the circular disk  $C$ .

### Results

$${}^L\mathbf{v}^P = r[3 \sin q_2(\dot{q}_2\mathbf{c}_1 - \dot{q}_1\mathbf{c}_2) + (\dot{q}_1 + 3\dot{q}_2 \cos q_2)\mathbf{c}_3]$$

$$\begin{aligned} {}^L\mathbf{a}^P &= r\{3(\ddot{q}_2 \sin q_2 + \dot{q}_2^2 \cos q_2)\mathbf{c}_1 \\ &\quad - (3\ddot{q}_1 \sin q_2 + 6\dot{q}_1\dot{q}_2 \cos q_2 + \dot{q}_1^2)\mathbf{c}_2 \\ &\quad + [\dot{q}_1 + 3\dot{q}_2 \cos q_2 - 3(\dot{q}_1^2 + \dot{q}_2^2) \sin q_2]\mathbf{c}_3\} \end{aligned}$$

$${}^C\mathbf{v}^P = 3r\dot{q}_2(\sin q_2\mathbf{c}_1 + \cos q_2\mathbf{c}_3)$$

$${}^C\mathbf{a}^P = 3r[(\ddot{q}_2 \sin q_2 + \dot{q}_2^2 \cos q_2)\mathbf{c}_1 + (\ddot{q}_2 \cos q_2 - \dot{q}_2^2 \sin q_2)\mathbf{c}_3]$$

3.2 A point  $P$  moves on a space curve  $C$  fixed in a reference frame  $A$ . Show that  $\mathbf{v}$ , the velocity of  $P$  in  $A$ , can be expressed as

$$\mathbf{v} = v\mathbf{b}_1$$

where  $\mathbf{b}_1$  is a vector tangent of  $C$  at  $P$  (see Problem 2.1), and verify that  $\mathbf{a}$ , the acceleration of  $P$  in  $A$ , is given by

$$\mathbf{a} = \dot{v}\mathbf{b}_1 + \frac{v^2}{\rho}\mathbf{b}_2$$

where  $\mathbf{b}_2$  and  $\rho$  are, respectively, the vector principal normal and the principal radius of curvature of  $C$  at  $P$ .

**3.3** Letting  $u_4 \triangleq \dot{q}_4$  and  $u_5 \triangleq \dot{q}_5$ , with  $q_4$  and  $q_5$  defined as in Problem 2.7, determine  $v_i$  ( $i = 1, 2, 3$ ) such that the velocity of  $C^*$  in  $A$  is given by

$${}^A\mathbf{v}^{C^*} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + v_3\mathbf{b}_3$$

Express  $v_i$  ( $i = 1, 2, 3$ ) as functions of  $q_j$  and  $u_j$  ( $j = 1, \dots, 5$ ).

### Results

$$v_1 = -Ru_2 \tan q_2 + u_4 c_1 + u_5 s_1$$

$$v_2 = (-u_4 s_1 + u_5 c_1)s_2$$

$$v_3 = Ru_1 + (u_4 s_1 - u_5 c_1)c_2$$

**3.4** If  $a_i$  is defined as  $a_i \triangleq {}^A\mathbf{a}^{C^*} \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), where  $A$  and  $C^*$  have the same meaning as in Problem 2.7 and  ${}^A\mathbf{a}^{C^*}$  denotes the acceleration of  $C^*$  in  $A$ , then  $a_1, a_2, a_3$  can be expressed as functions of  $q_j, u_j$ , and  $\dot{u}_j$  ( $j = 1, \dots, 5$ ), with  $u_4$  and  $u_5$  defined as in Problem 3.3. Determine these functions.

### Results

$$a_1 = -\dot{u}_2 R \tan q_2 + \dot{u}_4 c_1 + \dot{u}_5 s_1 + u_1 u_2 R(1 + \sec^2 q_2)$$

$$a_2 = (-\dot{u}_4 s_1 + \dot{u}_5 c_1)s_2 - Ru_1^2 - Ru_2^2 \tan^2 q_2$$

$$a_3 = \dot{u}_1 R + (\dot{u}_4 s_1 - \dot{u}_5 c_1)c_2 + u_2^2 R \tan q_2$$

**\*3.5** Determine the velocity  $\mathbf{v}$  and the acceleration  $\mathbf{a}$  of the midpoint of bar  $B_2$  of the linkage described in Problem 2.12 for time  $t^*$ .

$$\text{Results } \mathbf{v} = -24\beta_2 + 9\beta_3 \text{ m/s} \quad \mathbf{a} = 76\beta_2 + 93\beta_3 \text{ m/s}^2$$

**3.6** At time  $t$ , there exists precisely one point of the disk  $C$  of Problem 2.7 (see also Problems 2.8, 2.10, 3.3, and 3.4) that is in contact with the plane  $H$ . Calling this point  $\hat{C}$ , one can express the velocity and the acceleration of  $\hat{C}$  in  $A$  at time  $t$  as

$${}^A\mathbf{v}^{\hat{C}} = \hat{v}_x \mathbf{a}_x + \hat{v}_y \mathbf{a}_y$$

and

$${}^A\hat{\mathbf{a}} = \hat{a}_1\mathbf{b}_1 + \hat{a}_2\mathbf{b}_2 + \hat{a}_3\mathbf{b}_3$$

respectively, where  $\hat{v}_x$  and  $\hat{v}_y$  are functions of  $q_j$  and  $u_j$  ( $j = 1, \dots, 5$ ), and  $\hat{a}_i$  ( $i = 1, 2, 3$ ) are functions of  $q_j$ ,  $u_j$ , and  $\dot{u}_j$  ( $j = 1, \dots, 5$ ). Determine  $\hat{v}_x$ ,  $\hat{v}_y$ , and  $\hat{a}_i$  ( $i = 1, 2, 3$ ).

*Suggestion:* Use Eqs. (2.7.1) and (2.7.2), replacing the symbols  $P$ ,  $Q$ , and  $B$  with  $\hat{C}$ ,  $C^*$ , and  $C$ , respectively.

### Results

$$\hat{v}_x = (-u_2 \tan q_2 + u_3)Rc_1 + u_4$$

$$\hat{v}_y = (-u_2 \tan q_2 + u_3)Rs_1 + u_5$$

$$\hat{a}_1 = -\dot{u}_2 R \tan q_2 + \dot{u}_3 R + \dot{u}_4 c_1 + \dot{u}_5 s_1 + u_1 u_2 R \sec^2 q_2$$

$$\hat{a}_2 = (-\dot{u}_4 s_1 + \dot{u}_5 c_1)s_2 + R(u_3^2 - u_2^2 \tan^2 q_2)$$

$$\hat{a}_3 = (\dot{u}_4 s_1 - \dot{u}_5 c_1)c_2 + 2Ru_2(u_2 \tan q_2 - u_3)$$

**3.7** Suppose that the quantities  $\hat{v}_x$  and  $\hat{v}_y$  of Problem 3.6 are differentiated with respect to  $t$ , the results are used to form a vector  $\mathbf{a}$  defined as

$$\mathbf{a} \triangleq \frac{d\hat{v}_x}{dt} \mathbf{a}_x + \frac{d\hat{v}_y}{dt} \mathbf{a}_y$$

and quantities  $\bar{a}_i$  ( $i = 1, 2, 3$ ) then are formed as

$$\bar{a}_i \triangleq \mathbf{a} \cdot \mathbf{b}_i \quad (i = 1, 2, 3)$$

Do you expect  $\bar{a}_i$  to be equal to the quantity  $\hat{a}_i$  of Problem 3.6? First explain your answer, then establish its validity by determining  $\bar{a}_i$  ( $i = 1, 2, 3$ ).

**3.8** If  $\hat{A}$  is a point of a rigid body  $A$ ,  $\hat{B}$  a point of a rigid body  $B$ ,  $\hat{A}$  and  $\hat{B}$  are in contact with each other at time  $t$ , and the velocities of  $\hat{A}$  and  $\hat{B}$  in any reference frame are equal to each other at time  $t$ , then  $A$  and  $B$  are said to be *rolling* on each other at time  $t$ . Alternatively, one can say that *no slipping* is taking place at the contact between  $A$  and  $B$  at time  $t$ . (If  $A$  and  $B$  are in contact with each other at more than one point, these contacts must be considered separately. The bodies  $A$  and  $B$  can be rolling on each other at some points while slipping is taking place at other points.)

When the circular disk  $C$  of Problem 2.7 rolls on plane  $H$ , the quantities  $u_4$  and  $u_5$  of Problem 3.3 can be expressed as functions of  $q_j$  ( $j = 1, \dots, 5$ ) and  $u_k$  ( $k = 1, 2, 3$ ). Determine these functions.

**Results**  $u_4 = (u_2 \tan q_2 - u_3)Rc_1 \quad u_5 = (u_2 \tan q_2 - u_3)Rs_1$

**3.9** When the circular disk  $C$  of Problem 2.7 rolls on plane  $H$ , there exists at each instant one point  $\hat{C}$  of  $C$  that is in contact with  $H$ . The acceleration of  $\hat{C}$  in  $A$  is not, in general, equal to zero. Express the magnitude of this acceleration in terms of  $R$ ,  $u_2$ ,  $u_3$ , and  $\dot{q}_3$ .

**Result**  $R|\dot{q}_3|(u_2^2 + u_3^2)^{1/2}$

**3.10** When two rigid bodies  $A$  and  $B$  are rolling on each other (see Problem 3.8), the angular velocity  ${}^A\omega^B$  of  $B$  in  $A$  generally is not parallel to the plane  $P$  that is tangent to the surfaces of  $A$  and  $B$  at their points of contact with each other. When  ${}^A\omega^B$  is parallel to  $P$ , one speaks of *pure rolling* of  $A$  and  $B$  on each other.

Figure P3.10 shows a shaft terminating in a truncated cone  $C$  of semivertex angle  $\theta$  ( $0 < \theta < \pi/4$  rad), this shaft being supported by a thrust bearing consisting of a fixed race  $R$  and four identical spheres  $S$  of radius  $r$ . When the shaft rotates, rolling takes place at the two contacts between  $R$  and  $S$ , as well as at the contact between  $S$  and  $C$ . Moreover,  $S$  and  $C$  perform a pure rolling motion on each other (which is desirable because it minimizes wear) if the dimension  $b$  is a suitable function of  $r$  and  $\theta$ . Find this function.

**Result**  $r(1 + \sin \theta)/(\cos \theta - \sin \theta)$

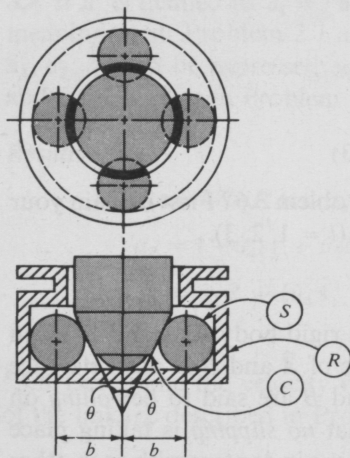


Figure P3.10

**3.11** The concept of rolling (see Problem 3.8) comes into play in connection with gearing, where it can be invoked in conjunction with Eqs. (2.7.1) and (2.4.1) to discover relationships between angular speeds (see Sec. 2.2) of gears.

Figure P3.11 shows schematically how the drive shaft  $D$  of an automobile can be connected to the two halves,  $A$  and  $A'$ , of an axle in such a way as to permit wheels attached rigidly to  $A$  and  $A'$  to rotate at different rates relative to the frame  $F$  that supports  $D$ ,  $A$ , and  $A'$ . This is accomplished as follows: Bevel gears  $B$



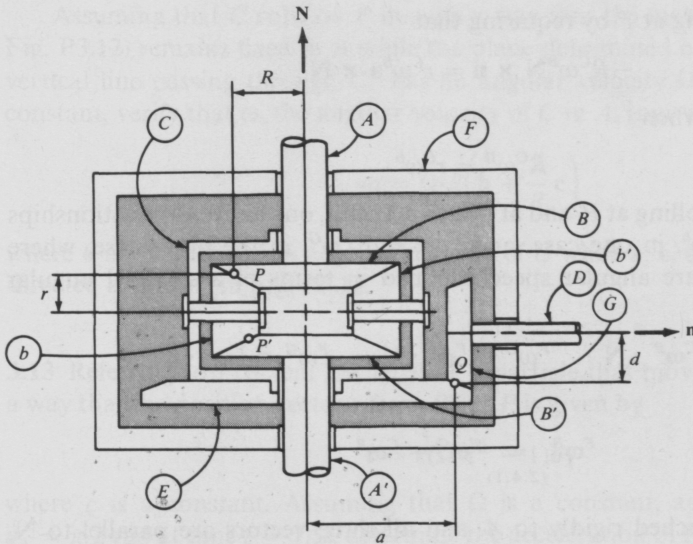


Figure P3.11

and  $B'$ , keyed to  $A$  and  $A'$ , respectively, engage bevel gears  $b$  and  $b'$ , respectively;  $b$  and  $b'$  are free to rotate on pins fixed in a casing  $C$  that can rotate about the (common) axis of  $A$  and  $A'$ , and a bevel gear  $E$ , fastened rigidly to  $C$ , is driven by a bevel gear  $G$  that is keyed to  $D$ .

Letting  $P$ ,  $P'$ , and  $Q$  be points selected arbitrarily on the lines of contact between  $B$  and  $b$ ,  $B'$  and  $b$ , and  $G$  and  $E$ , respectively, and assuming that rolling is taking place at these points, one can discover the relationship between angular speeds  ${}^F\omega^A$ ,  ${}^F\omega^{A'}$ , and  ${}^F\omega^D$ , defined as

$${}^F\omega^A \triangleq {}^F\omega^A \cdot \mathbf{N} \quad {}^F\omega^{A'} \triangleq {}^F\omega^{A'} \cdot \mathbf{N} \quad {}^F\omega^D \triangleq {}^F\omega^D \cdot \mathbf{n}$$

where  $\mathbf{N}$  and  $\mathbf{n}$  are unit vectors directed as shown in Fig. P3.11, by reasoning as follows.

$B$  and  $b$  have simple angular velocities in  $C$  and, after expressing these as

$${}^C\omega^B = {}^C\omega^B \mathbf{N} \quad {}^C\omega^b = {}^C\omega^b \mathbf{n}$$

one can let  $\hat{B}$  and  $\hat{b}$  be the points of  $B$  and  $b$ , respectively, that come into contact at  $P$ ; introduce distances  $R$  and  $r$  as shown in Fig. P3.11; write for the velocities of  $\hat{B}$  and  $\hat{b}$  in  $C$

$$\mathbf{v}_{\hat{B}} = {}^C\omega^B \times (-R\mathbf{n}) = -R {}^C\omega^B \mathbf{N} \times \mathbf{n} \quad (2.7.1)$$

and

$$\mathbf{v}_{\hat{b}} = {}^C\omega^b \times (r\mathbf{N}) = r {}^C\omega^b \mathbf{n} \times \mathbf{N} \quad (2.7.1)$$

and then ensure rolling at  $P$  by requiring that

$$-R^C \omega^B \mathbf{N} \times \mathbf{n} = r^C \omega^b \mathbf{n} \times \mathbf{N}$$

which is guaranteed when

$$R^C \omega^B = r^C \omega^b$$

Similarly, by taking rolling at  $P'$  and at  $Q$  into account, one arrives at relationships between  ${}^C \omega^{B'}$ , and  ${}^C \omega^b$ , in one case, and  ${}^F \omega^E$  and  ${}^F \omega^G$ , in the other case, where  ${}^C \omega^{B'}$ ,  ${}^F \omega^E$ , and  ${}^F \omega^G$  are angular speeds defined in terms of associated angular velocities as

$${}^C \omega^{B'} \triangleq {}^C \omega^{B'} \cdot \mathbf{N} \quad {}^F \omega^E \triangleq {}^F \omega^E \cdot \mathbf{N} \quad {}^F \omega^G \triangleq {}^F \omega^G \cdot \mathbf{n}$$

Next, one has

$${}^F \omega^B = {}^F \omega^C + {}^C \omega^B \quad (2.4.1)$$

which, since  $B$  is attached rigidly to  $A$ , and all three vectors are parallel to  $\mathbf{N}$ , implies that

$${}^F \omega^A = {}^F \omega^C + {}^C \omega^B$$

where  ${}^F \omega^C \triangleq {}^F \omega^C \cdot \mathbf{N}$ . Using Eq. (2.4.1) once more, this time in connection with  $B'$ ,  $C$ , and  $F$ , then brings one into position to find the relationship between  ${}^F \omega^A$ ,  ${}^F \omega^{A'}$ , and  ${}^F \omega^D$  by purely algebraic means.

Show that

$${}^F \omega^D = \frac{a}{2d} ({}^F \omega^A + {}^F \omega^{A'})$$

**3.12** Figure P3.12 shows a right-circular, uniform, solid cone  $C$  in contact with a horizontal plane  $P$  that is fixed in a reference frame  $A$ . The base of  $C$  has a radius  $R$ , and  $C$  has a height  $4h$ .

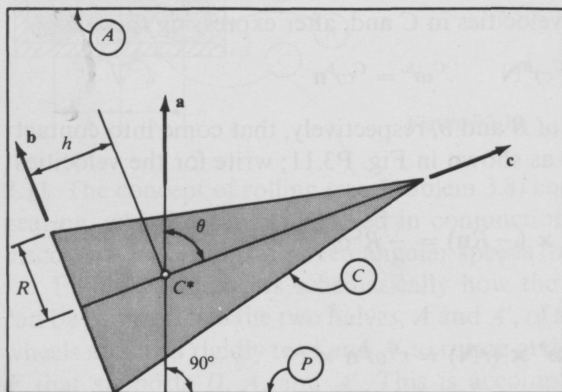


Figure P3.12

Assuming that  $C$  rolls on  $P$  in such a way that the mass center  $C^*$  of  $C$  (see Fig. P3.12) remains fixed in  $A$  while the plane determined by the axis of  $C$  and a vertical line passing through  $C^*$  has an angular velocity  $\Omega \mathbf{a}$  in  $A$ , where  $\Omega$  is a constant, verify that  $\omega$ , the angular velocity of  $C$  in  $A$ , is given by

$$\omega = \Omega s \theta \left( \mathbf{b} + \frac{h}{R} \mathbf{c} \right)$$

where  $\theta$  is the inclination angle of the axis of  $C$  while  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are unit vectors directed as shown in Fig. P3.12.

**3.13** Referring to Problem 2.9, let  $P$  be a particle that moves on body  $B$  in such a way that the position vector  $\mathbf{r}$  from  $B^*$  to  $P$  is given by

$$\mathbf{r} = c(\Omega^2 t^2 - 1)\mathbf{b}_3$$

where  $c$  is a constant. Assuming that  $\Omega$  is a constant, and letting  $q_1 = q_2 = q_3 = \pi/2$  rad at time  $t = 1/\Omega$ , determine the acceleration of  $P$  in  $N$  for this instant.

**Result**  $4c\Omega\dot{q}_1\mathbf{b}_1 - \Omega(R\Omega + 4c\dot{q}_3)\mathbf{b}_2 + 2c\Omega^2\mathbf{b}_3$

**3.14** The path point  $P$  mentioned in Problem 2.7 has a velocity both in  $A$  and in  $C$ , and these velocities can be expressed as

$${}^A\mathbf{v}^P = \tilde{v}_x \mathbf{a}_x + \tilde{v}_y \mathbf{a}_y + \tilde{v}_z \mathbf{a}_z$$

and

$${}^C\mathbf{v}^P = \tilde{v}_1 \mathbf{b}_1 + \tilde{v}_2 \mathbf{b}_2 + \tilde{v}_3 \mathbf{b}_3$$

respectively, where  $\tilde{v}_x$ ,  $\tilde{v}_y$ ,  $\tilde{v}_z$ , and  $\tilde{v}_i$  ( $i = 1, 2, 3$ ) are functions of  $q_j$  and  $u_j$  ( $j = 1, \dots, 5$ ), with  $u_4$  and  $u_5$  defined as in Problem 3.3. Determine  $\tilde{v}_x$ ,  $\tilde{v}_y$ ,  $\tilde{v}_z$ ,  $\tilde{v}_i$ , and  $\tilde{a}_i$  ( $i = 1, 2, 3$ ), with  $\tilde{a}_i$  defined as

$$\tilde{a}_i \triangleq {}^C\mathbf{a}^P \cdot \mathbf{b}_i \quad (i = 1, 2, 3)$$

where  ${}^C\mathbf{a}^P$  is the acceleration of  $P$  in  $C$ .

**Suggestion:** To find  ${}^C\mathbf{v}^P$  and  ${}^C\mathbf{a}^P$ , use Eqs. (2.8.1) and (2.8.2), replacing  $B$  with  $C$ , and  $\tilde{B}$  with  $\hat{C}$  as defined in Problem 3.6.

**Results**

$$\begin{aligned} \tilde{v}_x &= u_4 & \tilde{v}_y &= u_5 & \tilde{v}_z &= 0 \\ \tilde{v}_1 &= R(u_2 \tan q_2 - u_3) & \tilde{v}_2 &= \tilde{v}_3 &= 0 \\ \tilde{a}_1 &= R(\dot{u}_2 \tan q_2 - \dot{u}_3 - u_1 u_2 \sec^2 q_2) \\ \tilde{a}_2 &= R(u_2 \tan q_2 - u_3)^2 \\ \tilde{a}_3 &= 0 \end{aligned}$$

**\*3.15** Figure P3.15 is a schematic representation of a robot arm consisting of three elements  $A$ ,  $B$ , and  $C$ , the last of which holds a rigid body  $D$  rigidly. One end of  $A$  is a hub that is made to rotate about a vertical axis fixed in the Earth  $E$ . At a point  $P$ ,  $B$  is connected to  $A$  by means of a motor (all parts of which are rigidly attached either to  $A$  or to  $B$ ) that causes  $B$  to rotate relative to  $A$  about a horizontal axis fixed in  $A$ , passing through  $P$ , and perpendicular to the axis of  $A$ . Finally,  $C$  is connected to  $B$  by means of a rack-and-pinion drive that can make  $C$  slide relative to  $B$ .

Letting  $L_A$ ,  $L_B$ , and  $L_P$  denote distances as indicated in Fig. P3.15, where  $A^*$ ,  $B^*$ , and  $C^*$  are points fixed on  $A$ ,  $B$ , and  $C$ , respectively, introduce unit vectors  $\mathbf{a}_i$  and  $\mathbf{b}_i$  ( $i = 1, 2, 3$ ) as shown in Fig. P3.15, let  $p_i \triangleq \mathbf{p}^{C^*D^*} \cdot \mathbf{b}_i$ , where  $\mathbf{p}^{C^*D^*}$  is the position vector from  $C^*$  to  $D^*$ , the mass center of  $D$ , and define  $u_1, u_2, u_3$  as

$$u_1 \triangleq {}^E\boldsymbol{\omega}^A \cdot \mathbf{a}_1 \quad u_2 \triangleq {}^A\boldsymbol{\omega}^B \cdot \mathbf{b}_2 \quad u_3 \triangleq {}^B\mathbf{v}^{C^*} \cdot \mathbf{b}_3$$

where  ${}^E\boldsymbol{\omega}^A$  is the angular velocity of  $A$  in  $E$ ,  ${}^A\boldsymbol{\omega}^B$  is the angular velocity of  $B$  in  $A$ , and  ${}^B\mathbf{v}^{C^*}$  is the velocity of  $C^*$  in  $B$ . Denoting the radian measure of the angle between the axes of  $A$  and  $B$  by  $q_1$ , letting  $s_1 \triangleq \sin q_1$ ,  $c_1 \triangleq \cos q_1$ , and designating as  $q_2$  the distance from  $B^*$  to  $C^*$ , determine  ${}^E\boldsymbol{\omega}^A$ ,  ${}^E\boldsymbol{\omega}^B$ ,  ${}^E\boldsymbol{\omega}^C$ , and  ${}^E\boldsymbol{\omega}^D$ , the angular velocities of  $A$ ,  $B$ ,  $C$ , and  $D$  in  $E$ ;  ${}^E\boldsymbol{\alpha}^A$ ,  ${}^E\boldsymbol{\alpha}^B$ ,  ${}^E\boldsymbol{\alpha}^C$ , and  ${}^E\boldsymbol{\alpha}^D$ , the angular accelerations of  $A$ ,  $B$ ,  $C$ , and  $D$  in  $E$ ;  ${}^E\mathbf{v}^{A^*}$ ,  ${}^E\mathbf{v}^{B^*}$ ,  ${}^E\mathbf{v}^{C^*}$ , and  ${}^E\mathbf{v}^{D^*}$ , the velocities of  $A^*$ ,  $B^*$ ,  $C^*$ , and  $D^*$  in  $E$ ; and  ${}^E\mathbf{a}^{A^*}$ ,  ${}^E\mathbf{a}^{B^*}$ ,  ${}^E\mathbf{a}^{C^*}$ , and  ${}^E\mathbf{a}^{D^*}$ , the accelerations of  $A^*$ ,  $B^*$ ,  $C^*$ , and  $D^*$  in  $E$ . (Note that  $\dot{q}_1 = u_2$  and  $\dot{q}_2 = u_3$ .)

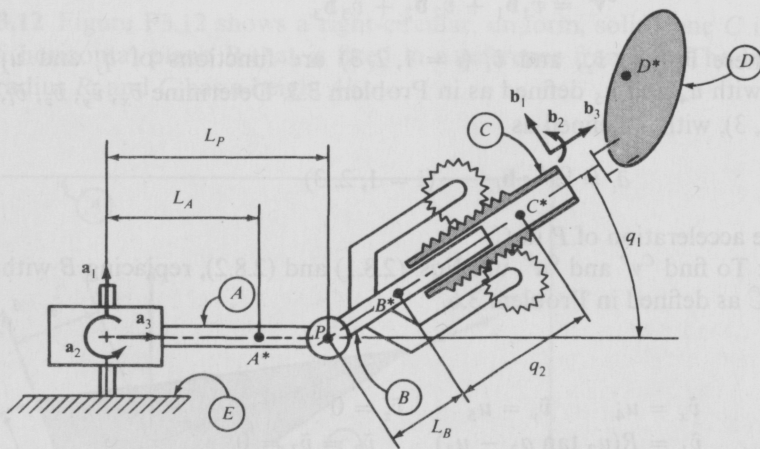


Figure P3.15

*Suggestion:* To facilitate the writing of results, introduce  $Z_1, \dots, Z_{34}$  as

$$\begin{aligned}
 Z_1 &\triangleq u_1 c_1 & Z_2 &\triangleq u_1 s_1 & Z_3 &\triangleq -Z_2 u_2 & Z_4 &\triangleq Z_1 u_2 & Z_5 &\triangleq -L_A u_1 \\
 Z_6 &\triangleq -(L_P + L_B c_1) & Z_7 &\triangleq u_2 L_B & Z_8 &\triangleq Z_6 u_1 & Z_9 &\triangleq L_B + q_2 \\
 Z_{10} &\triangleq Z_6 - q_2 c_1 & Z_{11} &\triangleq u_2 Z_9 & Z_{12} &\triangleq Z_{10} u_1 & Z_{13} &\triangleq -s_1 p_2 \\
 Z_{14} &\triangleq Z_9 + p_3 & Z_{15} &\triangleq Z_{10} + s_1 p_1 - c_1 p_3 & Z_{16} &\triangleq c_1 p_2 \\
 Z_{17} &\triangleq Z_{13} u_1 + Z_{14} u_2 & Z_{18} &\triangleq Z_{15} u_1 & Z_{19} &\triangleq Z_{16} u_1 - u_2 p_1 + u_3 \\
 Z_{20} &\triangleq u_1 Z_5 & Z_{21} &\triangleq L_B s_1 u_2 & Z_{22} &\triangleq -Z_2 Z_8 & Z_{23} &\triangleq Z_{21} u_1 + Z_2 Z_7 \\
 Z_{24} &\triangleq Z_1 Z_8 - u_2 Z_7 & Z_{25} &\triangleq Z_{21} - u_3 c_1 + q_2 s_1 u_2 \\
 Z_{26} &\triangleq 2u_2 u_3 - Z_2 Z_{12} & Z_{27} &\triangleq Z_{25} u_1 + Z_2 Z_{11} - Z_1 u_3 \\
 Z_{28} &\triangleq Z_1 Z_{12} - u_2 Z_{11} & Z_{29} &\triangleq -Z_{16} u_2 & Z_{30} &\triangleq Z_{25} + u_2 (c_1 p_1 + s_1 p_3) \\
 Z_{31} &\triangleq Z_{13} u_2 & Z_{32} &\triangleq Z_{29} u_1 + u_2 (u_3 + Z_{19}) - Z_2 Z_{18} \\
 Z_{33} &\triangleq Z_{30} u_1 + Z_2 Z_{17} - Z_1 Z_{19} & Z_{34} &\triangleq Z_{31} u_1 + Z_1 Z_{18} - u_2 Z_{17}
 \end{aligned}$$

### Results

$$\begin{aligned}
 {}^E \omega^A &= u_1 \mathbf{a}_1 \\
 {}^E \omega^B &= {}^E \omega^C = {}^E \omega^D = u_1 c_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_1 s_1 \mathbf{b}_3 = Z_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + Z_2 \mathbf{b}_3 \\
 {}^E \alpha^A &= \dot{u}_1 \mathbf{a}_1 & {}^E \alpha^B &= {}^E \alpha^C = {}^E \alpha^D = (\dot{u}_1 c_1 + Z_3) \mathbf{b}_1 + \dot{u}_2 \mathbf{b}_2 + (\dot{u}_1 s_1 + Z_4) \mathbf{b}_3 \\
 {}^E \mathbf{v}^{A*} &= -L_A u_1 \mathbf{a}_2 = Z_5 \mathbf{a}_2 & {}^E \mathbf{v}^{B*} &= u_2 L_B \mathbf{b}_1 + Z_6 u_1 \mathbf{b}_2 = Z_7 \mathbf{b}_1 + Z_8 \mathbf{b}_2 \\
 {}^E \mathbf{v}^{C*} &= u_2 Z_9 \mathbf{b}_1 + Z_{10} u_1 \mathbf{b}_2 + u_3 \mathbf{b}_3 = Z_{11} \mathbf{b}_1 + Z_{12} \mathbf{b}_2 + u_3 \mathbf{b}_3 \\
 {}^E \mathbf{v}^{D*} &= (Z_{13} u_1 + Z_{14} u_2) \mathbf{b}_1 + Z_{15} u_1 \mathbf{b}_2 + (Z_{16} u_1 - u_2 p_1 + u_3) \mathbf{b}_3 \\
 &= Z_{17} \mathbf{b}_1 + Z_{18} \mathbf{b}_2 + Z_{19} \mathbf{b}_3 \\
 {}^E \mathbf{a}^{A*} &= -L_A \dot{u}_1 \mathbf{a}_2 + Z_{20} \mathbf{a}_3 \\
 {}^E \mathbf{a}^{B*} &= (\dot{u}_2 L_B + Z_{22}) \mathbf{b}_1 + (Z_6 \dot{u}_1 + Z_{23}) \mathbf{b}_2 + Z_{24} \mathbf{b}_3 \\
 {}^E \mathbf{a}^{C*} &= (\dot{u}_2 Z_9 + Z_{26}) \mathbf{b}_1 + (Z_{10} \dot{u}_1 + Z_{27}) \mathbf{b}_2 + (\dot{u}_3 + Z_{28}) \mathbf{b}_3 \\
 {}^E \mathbf{a}^{D*} &= (Z_{13} \dot{u}_1 + Z_{14} \dot{u}_2 + Z_{32}) \mathbf{b}_1 + (Z_{15} \dot{u}_1 + Z_{33}) \mathbf{b}_2 \\
 &\quad + (Z_{16} \dot{u}_1 - p_1 \dot{u}_2 + \dot{u}_3 + Z_{34}) \mathbf{b}_3
 \end{aligned}$$

Note that two expressions are given for the angular velocities of  $B$ ,  $C$ , and  $D$  in  $E$ , as well as for the velocities of  $A^*$ ,  $B^*$ ,  $C^*$ , and  $D^*$  in  $E$ . In each case, the quantities  $u_1$ ,  $u_2$ ,  $u_3$  appear explicitly in the first expression but are absent from the second, except in the case of  ${}^E \mathbf{v}^{C*}$ , where no simplification would result from replacing  $u_3$  with another symbol. There are two reasons for writing each angular velocity and each velocity in these two ways. The first will become apparent in connection with Problem 4.17; the second is that both versions come into play when one seeks to write expressions for angular accelerations of rigid bodies and accelerations of mass centers in such a way that  $\dot{u}_1$ ,  $\dot{u}_2$ , and  $\dot{u}_3$  appear explicitly.

# PROBLEM SET 4

(Secs. 2.9–2.15)

**4.1** Mutually perpendicular unit vectors  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{a}_3 \triangleq \mathbf{a}_1 \times \mathbf{a}_2$  are fixed in a reference frame  $A$ , and a unit vector  $\mathbf{b}$  is fixed in a rigid body  $B$ , one of whose points,  $O$ , is fixed in  $A$ .  $B$  is brought into a general orientation in  $A$ , after  $\mathbf{b}$  has been aligned with  $\mathbf{a}_1$ , by being subjected to successive rotations characterized by the vectors  $\theta_1 \mathbf{a}_1, \theta_2 \mathbf{a}_2$ , and  $\theta_3 \mathbf{a}_3$ , where  $\theta_1, \theta_2$ , and  $\theta_3$  are the radian measures of angles. A particle  $P$  is free to move on a line  $L$  that is fixed in  $B$ , passes through  $O$ , and is parallel to  $\mathbf{b}$ .

Letting  $\mathbf{p}$  be the position vector from  $O$  to  $P$ , and defining  $x_i$  as

$$x_i \triangleq \mathbf{p} \cdot \mathbf{a}_i \quad (i = 1, 2, 3)$$

show that  $P$  is guaranteed to remain on  $L$  when  $x_1, x_2$ , and  $x_3$  satisfy the holonomic constraint equations

$$x_1 s_3 - x_2 c_3 = 0 \quad x_3 c_2 c_3 + x_1 s_2 = 0$$

where  $s_i \triangleq \sin \theta_i, c_i \triangleq \cos \theta_i$  ( $i = 2, 3$ ).

Letting  $q$  be an arbitrary quantity, verify that the constraint equations are satisfied identically if  $x_1 = qc_2 c_3, x_2 = qc_2 s_3, x_3 = -qs_2$ .

Determine  $x_1, x_2$ , and  $x_3$  such that  $\mathbf{p} = q\mathbf{b}$ .

**4.2** Figure P4.2 shows a double pendulum consisting of two particles,  $P_1$  and  $P_2$ , supported by rods of lengths  $L_1$  and  $L_2$  in such a way that the position vectors,  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , of  $P_1$  and  $P_2$  relative to a point  $O$  fixed in a reference frame  $A$  are at all times perpendicular to  $\mathbf{a}_z$ , one of three mutually perpendicular unit vectors,  $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ , fixed in  $A$ . Letting

$$x_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_x \quad y_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_y \quad z_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_z \quad (i = 1, 2)$$

construct functions  $f_j(x_1, y_1, z_1, x_2, y_2, z_2)$  ( $j = 1, \dots, 4$ ) such that four holonomic constraint equations governing motions of  $P_1$  and  $P_2$  in  $A$  can be expressed as  $f_j = 0$  ( $j = 1, \dots, 4$ ).

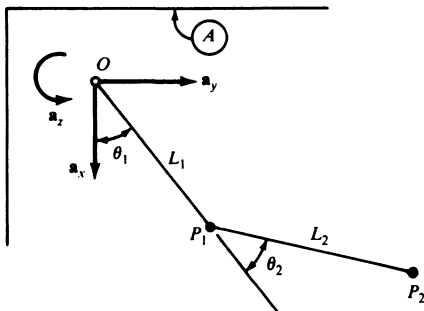


Figure P4.2

### Results

$$\begin{aligned} f_1 &= x_1^2 + y_1^2 - L_1^2 \\ f_2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 \\ f_3 &= z_1 \quad f_4 = z_2 \end{aligned}$$

**4.3** Figure P4.3 shows two particles,  $P_1$  and  $P_2$ , supported by a linkage in such a way that the position vectors,  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , of  $P_1$  and  $P_2$  relative to a point  $O$  fixed in a reference frame  $A$  are at all times perpendicular to  $\mathbf{a}_z$ , one of three mutually perpendicular unit vectors  $\mathbf{a}_x$ ,  $\mathbf{a}_y$ ,  $\mathbf{a}_z$ , fixed in  $A$ . Determine the number of holonomic constraint equations governing motions of  $P_1$  and  $P_2$  in  $A$ .

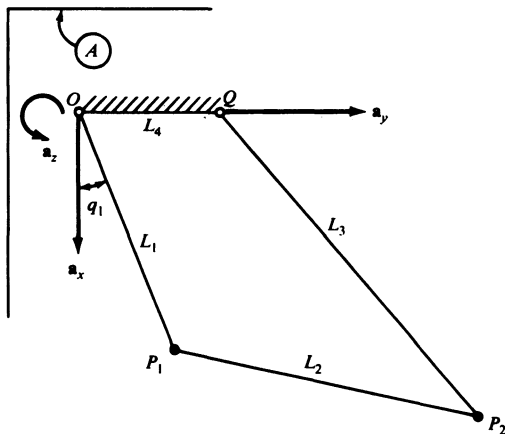


Figure P4.3

### Result 5

**4.4** Referring to Problem 4.1, and assuming that  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  are prescribed functions of time, show that only one generalized coordinate is required to specify the configuration of  $P$  in  $A$ , and verify that  $q$  is such a generalized coordinate.

**4.5** Referring to Problem 4.2, suppose that  $q_1$  and  $q_2$  are introduced by expressing  $x_i$ ,  $y_i$ , and  $z_i$  ( $i = 1, 2$ ) as

$$\begin{aligned} x_1 &= q_1 & y_1 &= (L_1^2 - q_1^2)^{1/2} & z_1 &= 0 \\ x_2 &= q_2 & y_2 &= (L_1^2 - q_1^2)^{1/2} + [L_2^2 - (q_2 - q_1)^2]^{1/2} & z_2 &= 0 \end{aligned}$$

Verify that the four holonomic constraint equations considered in Problem 4.2 then are satisfied. Explain the following statement: "For motions during which  $y_1$  and  $y_2$  acquire negative values,  $q_1$  and  $q_2$  are not generalized coordinates." Finally, letting  $\theta_1$  and  $\theta_2$  be the radian measures of two angles as indicated in Fig. P4.2, show that  $\theta_1$  and  $\theta_2$  are generalized coordinates for  $P_1$  and  $P_2$  in  $A$ .

4.6 Referring to Problem 4.3, let

$$x_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_x \quad y_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_y \quad z_i \triangleq \mathbf{p}_i \cdot \mathbf{a}_z \quad (i = 1, 2)$$

and, after noting that a single generalized coordinate suffices to characterize the configuration of  $P_1$  and  $P_2$  in  $A$ , attempt to express  $x_i$ ,  $y_i$ , and  $z_i$  ( $i = 1, 2$ ) as functions of the angle  $q_1$  shown in Fig. P4.3 in such a way that all holonomic constraint equations are satisfied. Next, after expressing  $x_1$ ,  $y_1$ , and  $z_1$  as

$$x_1 = L_1 c_1 \quad y_1 = L_1 s_1 \quad z_1 = 0$$

and  $x_2$ ,  $y_2$ , and  $z_2$  as

$$x_2 = L_1 c_1 + L_2 c_2 \quad y_2 = L_1 s_1 + L_2 s_2 \quad z_2 = 0$$

and as

$$x_2 = L_3 c_3 \quad y_2 = L_3 s_3 + L_4 \quad z_2 = 0$$

show that in both cases all holonomic constraint equations are satisfied provided that  $q_1$ ,  $q_2$ , and  $q_3$  satisfy the two equations

$$L_1 c_1 + L_2 c_2 - L_3 c_3 = 0$$

$$L_1 s_1 + L_2 s_2 - L_3 s_3 - L_4 = 0$$

Give a geometric interpretation of  $q_2$  and  $q_3$ , and explain the following statement: "Any one of  $q_1$ ,  $q_2$ , and  $q_3$ , but no more than one at a time, can be a generalized coordinate for  $P_1$  and  $P_2$  in  $A$ ."

4.7 Determine the number of generalized coordinates of each of the following systems in a reference frame  $A$ : (a) Two rigid bodies attached to each other by means of a ball-and-socket joint, but otherwise free to move in  $A$ . (b) A rigid body  $B$  carrying a rotor that is free to rotate relative to  $B$  about an axis fixed in  $B$  while  $B$  is free to move in  $A$ . (c) A rigid body  $B$  carrying a rotor that is made to rotate relative to  $A$  at a prescribed, time-dependent rate while  $B$  is free to move in  $A$ . (d) The system of two particles in Problem 4.2. (e) The system of two particles in Problem 4.3.

**Results** (a) 9; (b) 7; (c) 6; (d) 2; (e) 1

4.8 When the disk  $C$  of Problem 2.7 moves subject to the configuration constraint that  $C$  must remain in contact with  $H$ , then  $q_1, \dots, q_5$  are generalized coordinates for  $C$  in  $A$ . Show that  $u_1, \dots, u_5$ , defined as

$$u_i \triangleq {}^A\boldsymbol{\omega}^C \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \quad u_4 = {}^A\mathbf{v}^P \cdot \mathbf{a}_x \quad u_5 = {}^A\mathbf{v}^P \cdot \mathbf{a}_y$$

are generalized speeds for  $C$  in  $A$ , provided that  $|q_2| \neq \pi/2$  rad.



4.9 Referring to Problem 2.9, and defining generalized speeds  $u_1, u_2, u_3$  as (a)  $u_i \triangleq {}^N\boldsymbol{\omega}^B \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ) and (b)  $u_i \triangleq {}^A\boldsymbol{\omega}^B \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), determine  $Z_1, Z_2, Z_3$  such that Eqs. (2.12.1) are satisfied.

**Results**

$$\begin{aligned} (a) \quad & Z_1 = -\Omega c_1 s_2 \\ & Z_2 = \Omega(c_1 c_2 s_3 + c_3 s_1) \\ & Z_3 = \Omega(c_1 c_2 c_3 - s_3 s_1) \\ (b) \quad & Z_1 = Z_2 = Z_3 = 0 \end{aligned}$$

4.10 Referring to Problem 2.7, and considering only motions of rolling of  $C$  on  $H$ , let  $u_1, \dots, u_5$  be generalized speeds defined as in Problem 4.8. Determine  $A_{rs}$  and  $B_r$  ( $r = 4, 5; s = 1, 2, 3$ ) such that Eqs. (2.13.1) are satisfied.

**Results**

$$\begin{aligned} A_{41} &= 0 & A_{42} &= R \cos q_1 \tan q_2 & A_{43} &= -R \cos q_1 \\ A_{51} &= 0 & A_{52} &= R \sin q_1 \tan q_2 & A_{53} &= -R \sin q_1 \\ B_4 &= B_5 = 0 \end{aligned}$$

4.11 Figure P4.11 shows two sharp-edged circular disks,  $C_1$  and  $C_2$ , each of radius  $R$ , mounted at the extremities of a cylindrical shaft  $S$  of length  $2L$ , the axis of  $S$  coinciding with those of  $C_1$  and  $C_2$ . The disks are supported by a plane  $P$  that is fixed in a reference frame  $A$ , and they can rotate freely relative to  $S$ .

Six generalized speeds suffice to characterize all motions of  $C_1, C_2$ , and  $S$  in  $A$ . Defining these as

$$\begin{aligned} u_1 &\triangleq {}^A\mathbf{v}^{S^*} \cdot \mathbf{n}_1 & u_2 &\triangleq {}^A\boldsymbol{\omega}^S \cdot \mathbf{n}_2 & u_3 &\triangleq {}^A\boldsymbol{\omega}^S \cdot \mathbf{n}_3 \\ u_4 &\triangleq {}^A\boldsymbol{\omega}^{C_1} \cdot \mathbf{n}_3 & u_5 &\triangleq {}^A\boldsymbol{\omega}^{C_2} \cdot \mathbf{n}_3 & u_6 &\triangleq {}^A\mathbf{v}^{S^*} \cdot \mathbf{n}_3 \end{aligned}$$

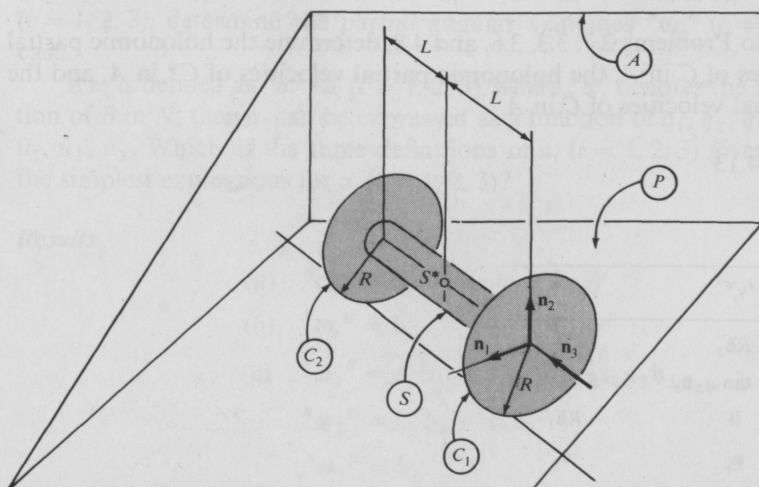


Figure P4.11

where  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular unit vectors, with  $\mathbf{n}_2$  normal to  $P$ , as shown in Fig. P4.11, and  $S^*$  is the mass center of  $S$ , show that when  $C_1$  and  $C_2$  roll on  $P$ , then  $u_4$ ,  $u_5$ , and  $u_6$  each can be expressed in terms of  $u_1$ ,  $u_2$ , and  $u_3$  by means of an equation having the form of Eq. (2.13.1). Determine  $A_{rs}$  and  $B_r$  ( $r = 4, 5, 6$ ;  $s = 1, 2, 3$ ).

**Results**

$$\begin{aligned} A_{41} &= -1/R & A_{42} &= L/R & A_{43} &= 0 \\ A_{51} &= -1/R & A_{52} &= -L/R & A_{53} &= 0 \\ A_{61} &= A_{62} = A_{63} = 0 \\ B_4 &= B_5 = B_6 = 0 \end{aligned}$$

**4.12** Referring to Problem 4.11, suppose that motors connecting  $S$  to  $C_1$  and  $C_2$  are used to cause  $C_1$  and  $C_2$  to rotate in such a way that

$${}^S\omega^{C_1} = \Omega_1 \mathbf{n}_3 \quad {}^S\omega^{C_2} = \Omega_2 \mathbf{n}_3$$

where  $\Omega_1$  and  $\Omega_2$  are prescribed functions of the time  $t$ . Show that the system formed by  $S$ ,  $C_1$ , and  $C_2$  possesses one degree of freedom in  $A$  if  $C_1$  and  $C_2$  roll on  $A$ , and determine  $A_{r1}$  and  $B_r$  ( $r = 2, \dots, 6$ ) such that  $u_2, \dots, u_6$  satisfy Eqs. (2.13.1).

**Results**

$$\begin{aligned} A_{21} &= 0 & B_2 &= \frac{1}{2}(R/L)(\Omega_1 - \Omega_2) \\ A_{31} &= -1/R & B_3 &= -\frac{1}{2}(\Omega_1 + \Omega_2) \\ A_{41} &= -1/R & B_4 &= \frac{1}{2}(\Omega_1 - \Omega_2) \\ A_{51} &= -1/R & B_5 &= \frac{1}{2}(\Omega_2 - \Omega_1) \\ A_{61} &= 0 & B_6 &= 0 \end{aligned}$$

**4.13** Referring to Problems 2.7, 3.3, 3.6, and 4.8, determine the holonomic partial angular velocities of  $C$  in  $A$ , the holonomic partial velocities of  $C^*$  in  $A$ , and the holonomic partial velocities of  $\hat{C}$  in  $A$ .

**Results** Table P4.13

**Table P4.13**

$r$	${}^A\omega_r^C$	${}^A\mathbf{v}_r^{C^*}$	${}^A\mathbf{v}_r^{\hat{C}}$
1	$\mathbf{b}_1$	$R\mathbf{b}_3$	0
2	$\mathbf{b}_2$	$-R \tan q_2 \mathbf{b}_1$	$-R \tan q_2 \mathbf{b}_1$
3	$\mathbf{b}_3$	0	$R\mathbf{b}_1$
4	0	$\mathbf{a}_x$	$\mathbf{a}_x$
5	0	$\mathbf{a}_y$	$\mathbf{a}_y$

**4.14** With  $u_1, \dots, u_5$  defined as in Problem 4.8, and considering only motions of rolling of  $C$  on  $H$ , determine the nonholonomic partial angular velocities of  $C$  in  $A$ , the nonholonomic partial velocities of  $C^*$  in  $A$ , and the nonholonomic partial velocities of  $\hat{C}$  in  $A$ . Do this by inspecting expressions for  ${}^A\omega^C$ ,  ${}^A\mathbf{v}^{C^*}$ , and  ${}^A\mathbf{v}^{\hat{C}}$  then check the results by using Eqs. (2.14.15) and (2.14.17) in conjunction with information available in Problems 4.10 and 4.13.

**Results** Table P4.14

**Table P4.14**

$r$	${}^A\omega_r^C$	${}^A\mathbf{v}_r^{C^*}$	${}^A\mathbf{v}_r^{\hat{C}}$
1	$\mathbf{b}_1$	$R\mathbf{b}_3$	0
2	$\mathbf{b}_2$	0	0
3	$\mathbf{b}_3$	$-R\mathbf{b}_1$	0

**\*4.15** The configuration of a system  $S$  in a reference frame  $A$  is characterized by generalized coordinates  $q_1, \dots, q_n$ . Taking  $u_r \triangleq \dot{q}_r$  ( $r = 1, \dots, n$ ), and letting  $\beta$  be a vector fixed in a rigid body  $B$  belonging to  $S$ , show that

$$\frac{{}^A\partial\beta}{\partial q_r} = \omega_r \times \beta \quad (r = 1, \dots, n)$$

where  $\omega_r$  is the  $r$ th partial angular velocity of  $B$  in  $A$ .

**4.16** Referring to Problem 2.9, and defining generalized speeds  $u_1, u_2, u_3$  as (a)  $u_i = {}^N\omega^B \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), (b)  $u_i \triangleq {}^A\omega^B \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), and (c)  $u_i = \dot{q}_i$  ( $i = 1, 2, 3$ ), determine the partial angular velocities  ${}^N\omega_r^B$  ( $r = 1, 2, 3$ ) in each case.

If  $\alpha_i$  is defined as  ${}^N\alpha^B \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), where  ${}^N\alpha^B$  denotes the angular acceleration of  $B$  in  $N$ , then  $\alpha_i$  can be expressed as a function of  $q_1, q_2, q_3, u_1, u_2, u_3$ , and  $\dot{u}_1, \dot{u}_2, \dot{u}_3$ . Which of the three definitions of  $u_i$  ( $i = 1, 2, 3$ ) given above leads to the simplest expressions for  $\alpha_i$  ( $i = 1, 2, 3$ )?

**Results**

- (a)  ${}^N\omega_r^B = \mathbf{b}_r \quad (r = 1, 2, 3)$
- (b)  ${}^N\omega_r^B = \mathbf{b}_r \quad (r = 1, 2, 3)$
- (c)  ${}^N\omega_1^B = c_2\mathbf{b}_1 + s_2s_3\mathbf{b}_2 + s_2c_3\mathbf{b}_3$   
 ${}^N\omega_2^B = c_3\mathbf{b}_2 - s_3\mathbf{b}_3$   
 ${}^N\omega_3^B = \mathbf{b}_1$

Definition (a) leads to the simplest expression for  $\alpha_i$  ( $i = 1, 2, 3$ ).

**\*4.17** Referring to Problem 3.15, determine the partial angular velocities  ${}^E\omega_r^A$ ,  ${}^E\omega_r^B$ ,  ${}^E\omega_r^C$ , and  ${}^E\omega_r^D$  ( $r = 1, 2, 3$ ) and the partial velocities  ${}^E\mathbf{v}_r^{A^*}$ ,  ${}^E\mathbf{v}_r^{B^*}$ ,  ${}^E\mathbf{v}_r^{C^*}$ , and  ${}^E\mathbf{v}_r^{D^*}$  ( $r = 1, 2, 3$ ).

**Results** Tables P4.17(a), P4.17(b)

**Table P4.17(a)**

$r$	${}^E\omega_r^A$	${}^E\omega_r^B$	${}^E\omega_r^C$	${}^E\omega_r^D$
1	$\mathbf{a}_1$	$c_1\mathbf{b}_1 + s_1\mathbf{b}_3$	$c_1\mathbf{b}_1 + s_1\mathbf{b}_3$	$c_1\mathbf{b}_1 + s_1\mathbf{b}_3$
2	0	$\mathbf{b}_2$	$\mathbf{b}_2$	$\mathbf{b}_2$
3	0	0	0	0

**Table P4.17(b)**

$r$	${}^E\mathbf{v}_r^{A^*}$	${}^E\mathbf{v}_r^{B^*}$	${}^E\mathbf{v}_r^{C^*}$	${}^E\mathbf{v}_r^{D^*}$
1	$-L_A\mathbf{a}_2$	$Z_6\mathbf{b}_2$	$Z_{10}\mathbf{b}_2$	$Z_{13}\mathbf{b}_1 + Z_{15}\mathbf{b}_2 + Z_{16}\mathbf{b}_3$
2	0	$L_B\mathbf{b}_1$	$Z_9\mathbf{b}_1$	$Z_{14}\mathbf{b}_1 - p_1\mathbf{b}_3$
3	0	0	$\mathbf{b}_3$	$\mathbf{b}_3$

**4.18** In Problem 4.8, five generalized speeds are defined for  $C$  in  $A$ . An alternative set of generalized speeds can be introduced by defining  $u_1, \dots, u_5$  as

$$u_r \triangleq \dot{q}_r \quad (r = 1, \dots, 5)$$

When  $C$  rolls on  $H$ , these generalized speeds satisfy constraint equations of the form of Eqs. (2.13.1), with  $A_{43} = -R \cos q_1$ ,  $A_{53} = -R \sin q_1$ , and  $A_{rs} = B_r = 0$  for  $r = 4, 5$  and  $s = 1, 2$ . Furthermore, the velocity of  $C^*$  in  $A$  is given by

$${}^A\mathbf{v}^{C^*} = -R[(u_1 \sin q_2 + u_3)\mathbf{b}_1 + u_2\mathbf{b}_3]$$

To verify that it can be far more laborious to work with the right-hand members of Eqs. (2.15.7) than with the left-hand members, determine  ${}^A\tilde{\mathbf{v}}_r^{C^*} \cdot {}^A\mathbf{a}^{C^*}$  ( $r = 1, 2, 3$ ), where  ${}^A\mathbf{a}^{C^*}$  is the acceleration of  $C^*$  in  $A$ , by (a) forming  ${}^A\tilde{\mathbf{v}}_r^{C^*}$  by inspection of the expression given above for  ${}^A\mathbf{v}^{C^*}$ , forming  ${}^A\mathbf{a}^{C^*}$  by differentiating  ${}^A\mathbf{v}^{C^*}$  with respect to time  $t$  in  $A$ , and then dot-multiplying  ${}^A\tilde{\mathbf{v}}_r^{C^*}$  with  ${}^A\mathbf{a}^{C^*}$  ( $r = 1, 2, 3$ ), and (b) using the right-hand members of Eqs. (2.15.7).

### Results

$${}^A\bar{v}_1^{C^*} \cdot {}^A\mathbf{a}^{C^*} = R^2(\dot{u}_1 \sin q_2 + 2u_1 u_2 \cos q_2 + \dot{u}_3) \sin q_2$$

$${}^A\bar{v}_2^{C^*} \cdot {}^A\mathbf{a}^{C^*} = R^2[\dot{u}_2 - u_1 \cos q_2(u_1 \sin q_2 + u_3)]$$

$${}^A\bar{v}_3^{C^*} \cdot {}^A\mathbf{a}^{C^*} = R^2(\dot{u}_1 \sin q_2 + 2u_1 u_2 \cos q_2 + \dot{u}_3)$$

## PROBLEM SET 5

(Secs. 3.1–3.5)

**5.1** Regarding Fig. P5.1 as showing two views of a body  $B$  formed by matter distributed uniformly (a) over a surface having no planar portions and (b) throughout a solid, determine (by integration) the coordinates  $x^*$ ,  $y^*$ ,  $z^*$  of the mass center of  $B$ .

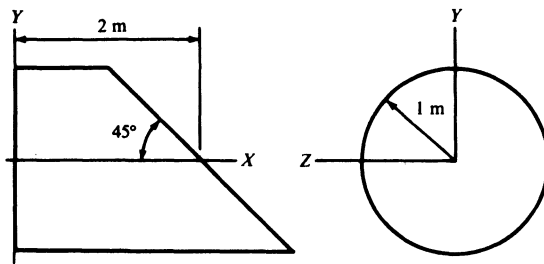


Figure P5.1

**Results** Table P5.1

Table P5.1

	$x^*$ (m)	$y^*$ (m)	$z^*$ (m)
(a)	$\frac{9}{8}$	$-\frac{1}{4}$	0
(b)	$\frac{17}{16}$	$-\frac{1}{8}$	0

**5.2** Regarding Fig. P5.2 as showing two views of a body  $B$  formed by matter distributed uniformly (a) over a surface having no planar portions and (b) throughout a solid, determine (without integration) the  $X$ -coordinate of the mass center of  $B$ .

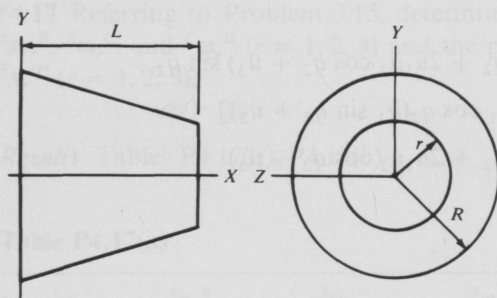


Figure P5.2

**Results** (a)  $\frac{L R + 2r}{3 R + r}$  (b)  $\frac{L R^2 + 2rR + 3r^2}{4 R^2 + rR + r^2}$

**\*5.3** Prove the following theorems (known as “theorems of Pappus” or “Guldin’s rules”):

When a plane curve  $C$  of length  $L$  is revolved about a line lying in the plane of  $C$  and not intersecting  $C$ , the area of the surface of revolution thus generated is equal to the product of  $L$  and the circumference of the circle described by the centroid of  $C$ .

When a plane region  $R$  of area  $A$  is revolved about a line lying in the plane of  $R$  and not passing through  $R$ , the volume of the solid of revolution thus generated is equal to the product of  $A$  and the circumference of the circle described by the centroid of  $R$ .

Use these theorems to locate the centroids of a semicircular curve and a semicircular sector, keeping in mind that the surface area and the volume of a sphere of radius  $R$  are equal to  $4\pi R^2$  and  $4\pi R^3/3$ , respectively.

**5.4** Parts  $A$ ,  $B$ ,  $C$ ,  $D$  of the assembly shown in Fig. P5.4 are made of steel ( $7800 \text{ kg/m}^3$ ), sheet metal ( $17.00 \text{ kg/m}^2$ ), aluminum ( $2700 \text{ kg/m}^3$ ), and brass ( $8400 \text{ kg/m}^3$ ),

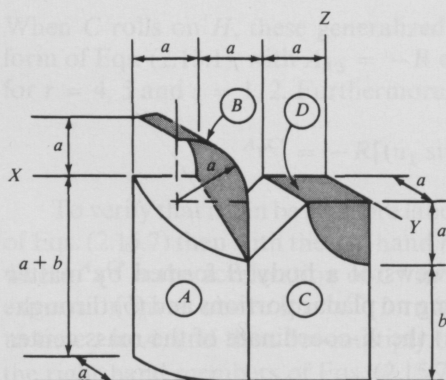


Figure P5.4

respectively. (a) For  $a = b = 0.3$  m, determine the coordinates  $x^*$ ,  $y^*$ ,  $z^*$  of the mass center of the assembly. (b) For  $a = 0.3$  m, determine to three significant figures the range of values of  $b$  such that  $x^* = 0.400$ .

**Results** (a) 0.434 m, 0.135 m,  $-0.300$  m (b)  $0.136 \text{ m} \leq b \leq 0.138 \text{ m}$

**5.5** Show by means of an example that  $\mathbf{I}_a$  as defined in Eq. (3.3.1) can be, but need not be, parallel to  $\mathbf{n}_a$ .

**5.6** Unit vectors  $\mathbf{n}_x$ ,  $\mathbf{n}_y$ , and  $\mathbf{n}_z$  are respectively parallel to the axes  $OX$ ,  $OY$ ,  $OZ$  of a rectangular Cartesian coordinate system, and each unit vector points in the positive direction of the axis to which it is parallel. Letting  $S$  be a set of  $v$  particles,  $m_i$  the mass of particle  $P_i$ , and  $x_i$ ,  $y_i$ , and  $z_i$  the coordinates of  $P_i$  ( $i = 1, \dots, v$ ), express  $I_x$ , the moment of inertia of  $S$  about the  $X$ -axis, and  $I_{yz}$ , the product of inertia of  $S$  relative to  $O$  for  $\mathbf{n}_y$  and  $\mathbf{n}_z$ , in terms of the masses and coordinates of  $P_1, \dots, P_v$ . Then answer the following questions: (a) Does it matter whether  $\mathbf{n}_x$ ,  $\mathbf{n}_y$ , and  $\mathbf{n}_z$  form a right-handed or a left-handed set of unit vectors? (b) Would the results be altered if  $OX$ ,  $OY$ , and  $OZ$  were not mutually perpendicular?

**Results**  $I_x = \sum_{i=1}^v m_i(y_i^2 + z_i^2)$   $I_{yz} = - \sum_{i=1}^v m_i y_i z_i$  (a) No (b) Yes

**5.7** Show by means of examples that products of inertia can be positive, negative, or zero, and that radii of gyration can be equal to zero.

**\*5.8** A body  $B$  of mass  $m$  is modeled as matter distributed uniformly along a helix  $H$  constructed by drawing a straight line  $L$  on a rectangular sheet of paper having the dimensions shown in Fig. P5.8 and then bending the paper to form a right-circular cylinder. Letting  $\mathbf{n}_1$  and  $\mathbf{n}_2$  be unit vectors directed as shown, determine  $\mathbf{I}_1$  and  $\mathbf{I}_2$ , the inertia vectors of  $B$  relative to the mass center of  $B$  for  $\mathbf{n}_1$  and  $\mathbf{n}_2$ , respectively.

**Results**  $\mathbf{I}_1 = ma^2 \mathbf{n}_1 + [mab/(2\pi)] \mathbf{n}_1 \times \mathbf{n}_2$   $\mathbf{I}_2 = (m/2)(a^2 + b^2/6) \mathbf{n}_2$

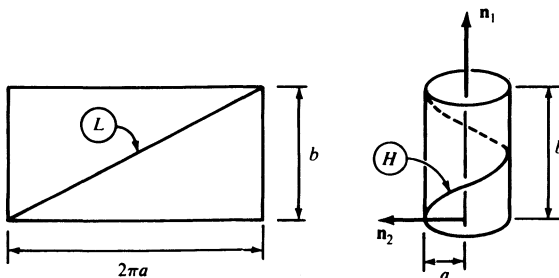


Figure P5.8

**5.9** A line  $L_a$  passes through a point  $O$  and is perpendicular to the unit vector  $\mathbf{n}_3$  of a set of mutually perpendicular unit vectors  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$ . Under these circumstances,  $I_a$ , the moment of inertia of a body  $B$  about line  $L_a$ , depends on the orientation of  $L_a$  in the plane that passes through  $O$  and is perpendicular to  $\mathbf{n}_3$ .

Express the maximum and minimum values of  $I_a$  in terms of  $I_1$ ,  $I_2$ , and  $I_{12}$ , the inertia scalars of  $B$  relative to  $O$  for  $\mathbf{n}_1$  and  $\mathbf{n}_2$ , and, letting  $\cos \theta \mathbf{n}_1 + \sin \theta \mathbf{n}_2$  be a unit vector parallel to  $L_a$ , show that

$$\tan 2\theta = \frac{2I_{12}}{I_1 - I_2}$$

when  $I_a$  has a maximum or minimum value.

**Result**

$$\frac{I_1 + I_2}{2} \pm \left[ \left( \frac{I_1 - I_2}{2} \right)^2 + I_{12}^2 \right]^{1/2}$$

**5.10** In Fig. P5.10,  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular unit vectors, and  $B^*$  designates the mass center of a body  $B$ . The inertia scalars of  $B$  relative to point  $O$  for  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are shown in units of  $\text{kg m}^2$  in Table P5.10.

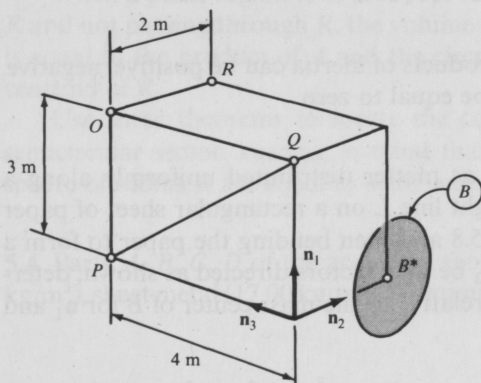


Figure P5.10

Table P5.10

	1	2	3
1	260	72	-144
2	72	325	96
3	-144	96	169

Determine the moment of inertia of  $B$  with respect to a line that is parallel to line  $PQ$  and passes through point  $O$ .

**Result**  $340 \text{ kg m}^2$

**5.11** Referring to Problem 5.10, and letting  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  be unit vectors defined as

$$\mathbf{e}_1 \triangleq -\mathbf{n}_3 \quad \mathbf{e}_2 \triangleq \mathbf{n}_1 \quad \mathbf{e}_3 \triangleq \mathbf{n}_2$$

form the inertia matrix of  $B$  relative to  $O$  for  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ .



**Result**

$$\begin{bmatrix} 169 & 144 & -96 \\ 144 & 260 & 72 \\ -96 & 72 & 325 \end{bmatrix}$$

**\*5.12** Solve Problem 5.10 by performing multiplications of a row matrix with the matrix constructed in Problem 5.11.

**5.13** If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  and  $\mathbf{n}_1', \mathbf{n}_2', \mathbf{n}_3'$  are two sets of unit vectors, the unit vectors of each set are mutually perpendicular, and  $C_{ij}$  is defined as

$$C_{ij} \triangleq \mathbf{n}_i \cdot \mathbf{n}_j' \quad (i, j = 1, 2, 3)$$

then the  $3 \times 3$  matrix  $C$  having  $C_{ij}$  as the  $j$ th element of the  $i$ th row called a *direction cosine matrix* for the two sets of unit vectors. Letting  $C^T$  denote the transpose of  $C$ , show that  $I$  and  $I'$ , the inertia matrices of a set  $S$  for a point  $O$  for  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  and  $\mathbf{n}_1', \mathbf{n}_2', \mathbf{n}_3'$ , respectively, are related to each other as follows:

$$I' = C^T I C$$

**\*5.14** The time-derivative of a dyadic  $\mathbf{D}$  in a reference frame  $A$  is defined as

$$\frac{{}^A d\mathbf{D}}{dt} \triangleq \sum_{i=1}^3 \sum_{j=1}^3 \mathbf{a}_i \mathbf{a}_j \frac{d}{dt} (\mathbf{a}_i \cdot \mathbf{D} \cdot \mathbf{a}_j)$$

where  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  are mutually perpendicular unit vectors fixed in  $A$ . Show that the time-derivatives of  $\mathbf{D}$  in two reference frames  $A$  and  $B$  are related by

$$\frac{{}^A d\mathbf{D}}{dt} = \frac{{}^B d\mathbf{D}}{dt} + {}^A \boldsymbol{\omega}^B \times \mathbf{D} - \mathbf{D} \times {}^A \boldsymbol{\omega}^B$$

where  ${}^A \boldsymbol{\omega}^B$  is the angular velocity of  $B$  in  $A$ .

**PROBLEM SET 6**

(Secs. 3.6–3.9)

**6.1** A point  $O$  of a rigid body  $B$  is fixed in a reference frame  $A$ . Show that  $\mathbf{H}$ , the angular momentum of  $B$  relative to  $O$  in  $A$ , is given by Eq. (3.5.28) if  $\mathbf{I}$  denotes the inertia dyadic of  $B$  relative to  $O$  and  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $A$ .

**\*6.2** A rigid body  $B$  moves in a reference frame  $A$  with an angular velocity  $\omega$ . Show that the central angular momentum of  $B$  in  $A$  is parallel to  $\omega$  if and only if  $\omega$  is parallel to a central principal axis of  $B$ .

**6.3** Letting  $\mathbf{H}^{S/O}$ ,  $\mathbf{H}^{S/S^*}$ , and  $\mathbf{H}^{S^*/O}$  denote, respectively, the angular momentum of a set  $S$  of particles relative to a point  $O$  in a reference frame  $A$ , the central angular momentum of  $S$  in  $A$ , and the angular momentum in  $A$ , relative to  $O$ , of a particle whose motion is identical to that of the mass center  $S^*$  of  $S$  and whose mass is equal to the total mass of  $S$ , show that

$$\mathbf{H}^{S/O} = \mathbf{H}^{S/S^*} + \mathbf{H}^{S^*/O}$$

**\*6.4** Letting  ${}^E\mathbf{H}^{C/C^*}$ ,  ${}^E\mathbf{H}^{D/C^*}$ , and  ${}^A\mathbf{H}^{B/C^*}$  denote angular momenta, show that

$${}^E\mathbf{H}^{C/C^*} = {}^E\mathbf{H}^{D/C^*} + {}^A\mathbf{H}^{B/C^*}$$

if  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $C^*$  are defined as follows:  $A$ , a rigid body;  $B$ , a set of particles;  $C$ , the system formed by  $A$  and  $B$ ;  $D$ , a rigid body that has the same motion as  $A$  and the same mass distribution as  $C$ ;  $E$ , a reference frame;  $C^*$ , the mass center of  $C$ .

**\*6.5** The mass center of a rigid body  $B$  is fixed in a rigid body  $A$ , but  $B$  is otherwise free to move relative to  $A$ . Letting  $C$  be the system formed by  $A$  and  $B$ , show that  $\mathbf{H}$ , the central angular momentum of  $C$  in a reference frame  $E$ , is given by

$$\mathbf{H} = \mathbf{I}_B \cdot {}^A\omega^B + \mathbf{I}_C \cdot {}^E\omega^A$$

where  $\mathbf{I}_B$  and  $\mathbf{I}_C$  are the central inertia dyadics of  $B$  and  $C$ , respectively, while  ${}^A\omega^B$  and  ${}^E\omega^A$  are the angular velocities of  $B$  in  $A$ , and  $A$  in  $E$ , respectively.

**6.6** The body  $B$  of Problem 5.10 has a mass of 12 kg. Determine the moment of inertia of  $B$  about line  $PQ$ , and find the product of inertia of  $B$  relative to  $B^*$  for  $\mathbf{n}_1$  and  $\mathbf{n}_2$ .

**Results** 3316/25 kg m<sup>2</sup>; 0

**6.7** Three identical uniform, square plates, each of mass  $m$ , are attached to each other as shown in Fig. P6.7. Determine the value of  $\theta$  for which the radius of gyration of this assembly with respect to line  $L$  has a minimum value, and find this value.

**Results** 340°; 0.696b

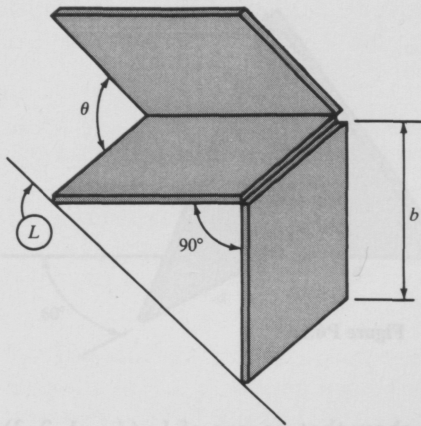


Figure P6.7

**6.8** A thin-walled, right-circular, cylindrical shell has a radius  $R$  and height  $H$ . Determine the radius of gyration of the shell with respect to a line that passes through the mass center and is perpendicular to the axis of the shell.

**Result**  $(R^2/2 + H^2/12)^{1/2}$

**6.9** Verify each of the following statements and provide an illustrative example: (a) A central principal axis of a body is a principal axis for each point of the axis. (b) If a principal axis for a point other than the mass center passes through the mass center, it is a central principal axis. (c) A line that is a principal axis for two of its points is a central principal axis. (d) The three principal axes for a point on a central principal axis are parallel to central principal axes. (e) If two principal moments of inertia for a given point are equal to each other, then the moments of inertia with respect to all lines passing through this point and lying in the plane determined by the associated principal axes are equal to each other. (f) If the particles of a set  $S$  lie in a plane  $P$ , then the line  $L$  normal to  $P$  and intersecting  $P$  at a point  $O$  is a principal axis of  $S$  for  $O$ , and the moment of inertia of  $S$  about  $L$  is equal to the sum of the moments of inertia of  $S$  about any two orthogonal lines that lie in  $P$  and intersect at  $O$ .

**6.10** Determine the smallest angle between line  $AB$  and any principal axis for point  $A$  of the thin, uniform, rectangular plate represented by the shaded portion of Fig. P6.10.

**Result**  $30.02^\circ$

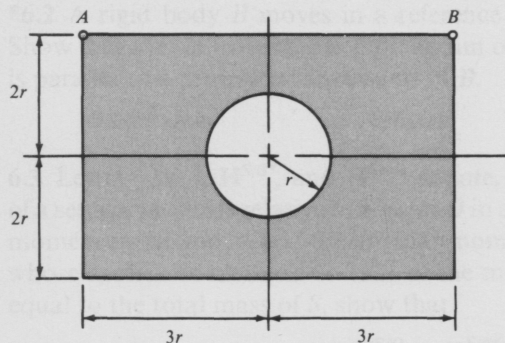


Figure P6.10

**6.11** For  $I_{jk}$  ( $j, k = 1, 2, 3$ ) as in Eq. (3.5.22), show that no one of  $I_{jj}$  ( $j = 1, 2, 3$ ) can exceed the sum of the other two, and that  $-I_{11}/2 \leq I_{23} \leq I_{11}/2$ ,  $-I_{22}/2 \leq I_{31} \leq I_{22}/2$ , and  $-I_{33}/2 \leq I_{12} \leq I_{33}/2$ .

**\*6.12** Defining  $C_1$ ,  $C_2$ , and  $C_3$  as

$$C_1 \triangleq I_1 + I_2 + I_3$$

$$C_2 \triangleq I_1 I_2 + I_2 I_3 + I_3 I_1 - I_{12}^2 - I_{23}^2 - I_{31}^2$$

$$C_3 \triangleq I_1 I_2 I_3 + 2I_{12} I_{23} I_{31} - I_1 I_{23}^2 - I_2 I_{31}^2 - I_3 I_{12}^2$$

where  $I_j \triangleq I_{jj}$  and  $I_{jk}$  is the inertia scalar of a set of particles relative to a point for unit vectors  $\mathbf{n}_j$  and  $\mathbf{n}_k$  ( $j, k = 1, 2, 3$ ), show that the values of  $C_1$ ,  $C_2$ , and  $C_3$  are independent of the way  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are chosen, so long as these vectors are mutually perpendicular.

*Suggestion:* Verify that Eq. (3.8.7) can be written

$$I_z^3 - C_1 I_z^2 + C_2 I_z - C_3 = 0$$

**6.13** Four identical particles are placed at the points  $O$ ,  $P$ ,  $Q$ ,  $R$  of Fig. P5.10. Determine the minimum radius of gyration of this set of particles, and find the smallest angle between the associated principal axis and line  $OP$ .

**Results** 1.436 m,  $67.64^\circ$

**6.14** Two identical, thin, uniform, right-triangular plates are attached to each other as shown in Fig. P6.14. When  $a/b$  is given,  $k$ , the minimum radius of gyration of this assembly, can be expressed as  $k = nb$ . Determine  $n$  for  $a/b = 2$  and  $a/b = 0.5$ .

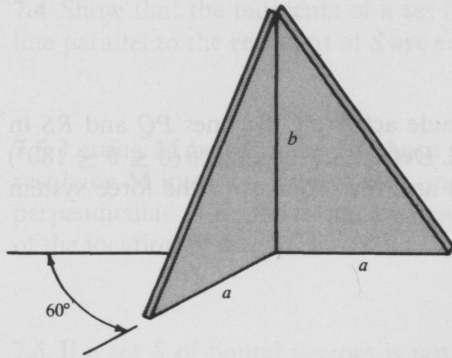


Figure P6.14

**Results**  $\frac{1}{3}, [35 - (241)^{1/2}]^{1/2}/24$

**\*6.15** Letting  $E$  be an inertia ellipsoid of a set  $S$  of particles for a point  $O$ , and letting  $R$  be the distance from  $O$  to a point  $P$  on the surface of  $E$ , show that  $\mathbf{I}_a$ , the inertia vector of  $S$  relative to  $O$  for  $\mathbf{n}_a$ , a unit vector directed from  $O$  to  $P$ , is perpendicular to  $E$  at  $P$ .

*Note:* If  $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$  are mutually perpendicular unit vectors, the position vector  $\mathbf{r}^{OP}$  from a point  $O$  to a point  $P$  of a surface  $\sigma$  is expressed as

$$\mathbf{r}^{OP} = x\mathbf{n}_x + y\mathbf{n}_y + z\mathbf{n}_z$$

and the equation of  $\sigma$  is written  $f(x, y, z) = 0$ , then  $\nabla f$ , a vector defined as

$$\nabla f \triangleq \frac{\partial f}{\partial x} \mathbf{n}_x + \frac{\partial f}{\partial y} \mathbf{n}_y + \frac{\partial f}{\partial z} \mathbf{n}_z$$

and called the *gradient* of  $f$ , is perpendicular to  $\sigma$  at  $P$ .

**\*6.16** For a set  $S$  of particles  $P_1, \dots, P_v$  moving in a reference frame  $A$ , a quantity  $G$ , known as a *Gibbs function* for  $S$  in  $A$ , is defined as

$$G \triangleq \frac{1}{2} \sum_{i=1}^v m_i \mathbf{a}_i^2$$

where  $m_i$  and  $\mathbf{a}_i$  are the mass of  $P_i$  and the acceleration of  $P_i$  in  $A$ , respectively.

Letting  $B$  be a rigid body, express the Gibbs function for  $B$  in  $A$  in terms of the acceleration  $\mathbf{a}$  of the mass center of  $B$  in  $A$ , the angular velocity  $\boldsymbol{\omega}$  of  $B$  in  $A$ , the angular acceleration  $\boldsymbol{\alpha}$  of  $B$  in  $A$ , the mass  $m$  of  $B$ , and the central inertia dyadic  $\mathbf{I}$  of  $B$ .

**Result**  $G = \frac{1}{2}(m\mathbf{a}^2 + \boldsymbol{\alpha} \cdot \mathbf{I} \cdot \boldsymbol{\alpha} + 2\boldsymbol{\alpha} \cdot \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega}^2 \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega})$

## PROBLEM SET 7

(Secs. 4.1–4.3)

7.1 Two forces, **A** and **B**, of equal magnitude act along the lines  $PQ$  and  $RS$  in Fig. P7.1 and are directed as there indicated. Determine the angle  $\theta$  ( $0 \leq \theta \leq 180^\circ$ ) between the resultant of this force system and the moment of the force system about point  $O$ .

**Result**  $117.12^\circ$

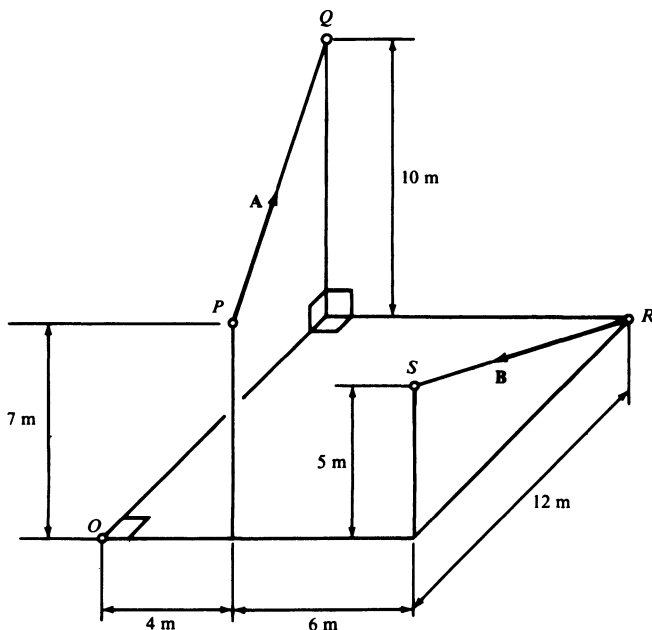


Figure P7.1

7.2 Referring to Problem 7.1, let **A** and **B** each have a magnitude of 10 N, and let **C** be a force applied at point  $O$  in such a way that **A**, **B**, and **C** form a couple. Determine the magnitude of the torque of this couple.

**Result** 90.56 N m

7.3 Show by means of an example that there exist sets of bound vectors such that the moment of the set of vectors about a point differs from the moment of the resultant about that point, no matter where the resultant is applied.

**7.4** Show that the moments of a set  $S$  of bound vectors about all points of any line parallel to the resultant of  $S$  are equal to each other.

**7.5** Letting  $\mathbf{M}$  be the moment about a point  $O$  of a set  $S$  of bound vectors, and resolving  $\mathbf{M}$  into two components, one parallel to  $\mathbf{R}$ , the resultant of  $S$ , the other perpendicular to  $\mathbf{R}$ , show that the magnitude of the first of these is independent of the location of  $O$ .

**7.6** If a set  $S$  of bound vectors is not a couple, the points about which  $S$  has a moment  $\mathbf{M}^*$  of minimum magnitude lie on a line  $L^*$  that is parallel to  $\mathbf{R}$ , the resultant of  $S$ .  $L^*$  is called the *central axis* of  $S$ .

Letting  $\mathbf{M}$  be the moment of  $S$  about a reference point  $O$  selected arbitrarily, show that

$$\mathbf{M}^* = \frac{\mathbf{R} \cdot \mathbf{M}}{R^2} \mathbf{R}$$

and that  $L^*$  passes through the point  $P^*$  such that the position vector  $\mathbf{p}^*$  from  $O$  to  $P^*$  is given by

$$\mathbf{p}^* = \frac{\mathbf{R} \times \mathbf{M}}{R^2}$$

(Note that  $\mathbf{p}^*$  is perpendicular to  $L^*$ ; thus, the distance from  $O$  to  $L^*$  is equal to  $|\mathbf{p}^*|$ .)

**7.7** Letting  $M$  denote the minimum of the magnitudes of all moments of the force system described in Problem 7.1, and  $R$  the magnitude of the resultant of this force system, determine  $M/R$  and find the distance from  $O$  to the central axis of the force system.

**Result** 6 m; 11.72 m

**7.8** A set of bound vectors consisting of a couple together with a single bound vector is called a *wrench* if the single bound vector is parallel to the torque of the couple. When the set of two forces of Problem 7.1 is replaced with a wrench consisting of a couple  $C$  together with a force  $\mathbf{F}$ , what is the magnitude of the torque of  $C$ , and what is the distance from point  $O$  to the line of action of  $\mathbf{F}$  if  $\mathbf{A}$  and  $\mathbf{B}$  each have a magnitude of 10 N?

*Suggestion:* Show that if the resultant of a set  $S$  of bound vectors is not equal to zero, then  $S$  can be replaced with a wrench consisting of the resultant of  $S$ ,

placed on the central axis of  $S$ , and a couple whose torque is equal to  $S$ 's moment of minimum magnitude.

**Results** 41.28 N m; 11.72 m

**7.9** Show that a set of bound vectors whose lines of action intersect at a point  $O$  can be replaced with the resultant of the set, applied at  $O$ .

**7.10** Show by means of an example that there exist sets of bound vectors, other than couples, that cannot be replaced with their resultants.

**7.11** Letting  $S$  be a set of bound vectors whose lines of action are coplanar and whose resultant is not equal to zero, show that  $S$  can be replaced with its resultant, placed on the central axis of  $S$ .

**7.12** Letting each of the forces in Problem 7.1 have a magnitude of 10 N, replace this set of forces with two forces  $A'$  and  $B'$ , with  $A'$  acting along line  $OP$ . Determine the magnitudes of  $A'$  and  $B'$ , and find the distance  $d$  from  $O$  to the line of action of  $B'$ .

**Results**  $|A'| = 4.961$  N;  $|B'| = 5.838$  N;  $d = 15.51$  m

**\*7.13** Let  $S$  be a set of coplanar forces  $F_i$  ( $i = 1, \dots, n$ ) applied at points  $P_i$  ( $i = 1, \dots, n$ ), respectively, and let  $\mathbf{p}_i$  be the position vector from a point  $O$  lying in the plane of the forces to  $P_i$ . Let  $S'$  be the set of forces  $F'_i$  ( $i = 1, \dots, n$ ) such that  $F'_i$  is obtained by rotating  $F_i$  about  $P_i$  through an angle  $\theta$  (in the plane of the forces). Then  $S$  can be replaced with a single force  $\mathbf{R}$ , and  $S'$  with a single force  $\mathbf{R}'$ , and the lines of action of  $\mathbf{R}$  and  $\mathbf{R}'$  intersect at a point  $Z$  called the *astatic center* of  $S$ .

Determine  $\mathbf{N}$  and  $v$  such that  $\mathbf{z}$ , the position vector from  $O$  to  $Z$ , can be expressed as

$$\mathbf{z} = \mathbf{R} \times \mathbf{N} + \mathbf{R}v$$

Express the results in terms of  $\mathbf{p}_i$  and  $F_i$  ( $i = 1, \dots, n$ ), and note that  $\mathbf{z}$  is independent of  $\theta$ .

*Suggestion:* Take advantage of the fact that if  $\mathbf{a}$  and  $\mathbf{b}$  are the position vectors from a point  $O$  to points  $A$  and  $B$ , respectively, and lines  $L_A$  and  $L_B$  lie in the plane  $P$  determined by  $O$ ,  $A$  and  $B$ , with  $L_A$  perpendicular to  $\mathbf{a}$  at  $A$ , and  $L_B$



perpendicular to  $\mathbf{b}$  at  $B$ , then  $\mathbf{c}$ , the position vector from  $O$  to  $C$ , the intersection of  $L_A$  and  $L_B$ , is given by

$$\mathbf{c} = \mathbf{a} + \frac{(\mathbf{b}^2 - \mathbf{a} \cdot \mathbf{b})\mathbf{a} \times \mathbf{k}}{\mathbf{a} \times \mathbf{k} \cdot \mathbf{b}}$$

where  $\mathbf{k}$  is a unit vector normal to  $P$ .

$$\mathbf{N} = \sum_{i=1}^n \mathbf{p}_i \times \mathbf{F}_i / \left( \sum_{i=1}^n \mathbf{F}_i \right)^2 \quad v = \sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{F}_i / \left( \sum_{i=1}^n \mathbf{F}_i \right)^2$$

## PROBLEM SET 8

(Secs. 4.4–4.11)

**8.1** Referring to the example in Sec. 2.9, let line  $Y$  be vertical, and let  $P_1$  and  $P_2$  have masses  $m_1$  and  $m_2$ , respectively. Define three sets of generalized speeds  $u_1, u_2, u_3$  as

$$\begin{aligned} u_1 &\triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{a}_x & u_2 &\triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{a}_y & u_3 &\triangleq \dot{q}_3 \\ u_1 &\triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_x & u_2 &\triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_y & u_3 &\triangleq \dot{q}_3 \\ u_1 &\triangleq \dot{q}_1 & u_2 &\triangleq \dot{q}_2 & u_3 &\triangleq \dot{q}_3 \end{aligned}$$

where  $\mathbf{a}_x$  and  $\mathbf{a}_y$  are unit vectors directed as shown in Fig. 2.9.2,  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are unit vectors directed as shown in Fig. 2.6.1, and  $q_1, q_2$ , and  $q_3$  are two distances and an angle, as indicated in Fig. 2.6.1. Assume that the panes of glass forming  $B$  are perfectly smooth, so that any forces they exert on  $P_1$  and  $P_2$  are parallel to  $\mathbf{b}_z$  in Fig. 2.6.1; and, regarding the mass of  $R$  as negligible, let the contact forces exerted by  $R$  on  $P_1$  and  $P_2$  be equal to  $C\mathbf{e}_x$  and  $-C\mathbf{e}_x$ , respectively, where  $C$  is a function of  $q_1, q_2, q_3, u_1, u_2, u_3$ .

Letting  $S$  be the set of two particles  $P_1$  and  $P_2$ , form expressions for the generalized active forces  $F_1, F_2, F_3$  for  $S$  in reference frame  $A$ , doing so for each of the three sets of generalized speeds defined above.

### Results

$$\begin{aligned} F_1 &= 0 & F_2 &= -(m_1 + m_2)g & F_3 &= -Lm_2gc_3 \\ F_1 &= -(m_1 + m_2)gs_3 & F_2 &= -(m_1 + m_2)gc_3 & F_3 &= -Lm_2gc_3 \\ F_1 &= 0 & F_2 &= -(m_1 + m_2)g & F_3 &= -Lm_2gc_3 \end{aligned}$$

**8.2** Referring to Problem 8.1, suppose that  $P_2$  is replaced with a sharp-edged circular disk  $D$  (of mass  $m_2$ ) whose axis is normal to the rod  $R$  and parallel to the plane in which  $R$  moves, as indicated in Fig. 2.13.1; further, that  $D$  comes into

contact with the two panes of glass at the points  $D_1$  and  $D_2$ . (The same assumptions were made in the example in Sec. 2.13.) Assume that  $D$  can be regarded as a particle on which the panes of glass exert contact forces equivalent to a force  $Y\mathbf{e}_y + Z\mathbf{e}_z$  applied at  $D^*$ . With  $u_1, u_2, u_3$  defined as

$$u_1 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_x \quad u_2 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_y \quad u_3 \triangleq \dot{q}_3$$

and letting  $S$  be the set of two particles  $P_1$  and  $D$ , determine (a) the generalized active forces  $F_1, F_2$ , and  $F_3$  for  $S$  in  $A$  and (b) the generalized active forces  $\tilde{F}_1$  and  $\tilde{F}_2$  for  $S$  in  $A$ . To find the results for (b), use Eqs. (4.4.1), then check the results by using Eqs. (4.4.3) together with the results from (a).

### Results

$$\begin{aligned} (a) \quad F_1 &= -(m_1 + m_2)gs_3 \\ F_2 &= -(m_1 + m_2)gc_3 + Y \\ F_3 &= -L(m_2gc_3 - Y) \\ (b) \quad \tilde{F}_1 &= -(m_1 + m_2)gs_3 \\ \tilde{F}_2 &= -m_1gc_3 \end{aligned}$$

**8.3** Referring to Problem 2.7, suppose that  $C$  is of uniform density, so that  $C^*$  is the mass center of  $C$ , and let  $m$  be the mass of  $C$ . Let  $P_x\mathbf{a}_x + P_y\mathbf{a}_y + P_z\mathbf{a}_z$  be the contact force exerted by  $H$  on  $C$  at point  $P$ , and introduce generalized speeds  $u_1, \dots, u_5$  as

$$\begin{aligned} u_i &\triangleq {}^A\boldsymbol{\omega}^C \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \\ u_4 &\triangleq \dot{q}_4 \quad u_5 \triangleq \dot{q}_5 \end{aligned}$$

Determine (a) the generalized active forces  $F_1, \dots, F_5$  for  $C$  in  $A$ , assuming that slipping is taking place at  $P$ , and (b) the generalized active forces  $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$  for  $C$  in  $A$ , assuming that  $C$  is rolling on  $H$ .

### Results

$$\begin{aligned} (a) \quad F_1 &= -Rmgs_2 \\ F_2 &= -R \tan q_2(c_1P_x + s_1P_y) \\ F_3 &= R(c_1P_x + s_1P_y) \\ F_4 &= P_x \\ F_5 &= P_y \\ (b) \quad \tilde{F}_1 &= -Rmgs_2 \\ \tilde{F}_2 &= 0 \\ \tilde{F}_3 &= 0 \end{aligned}$$

**8.4** Referring to Problem 4.11, suppose that  $P$  is horizontal,  $C_1$  and  $C_2$  roll on  $P$ , and motors connecting  $S$  to  $C_1$  and  $C_2$  cause  $C_1$  and  $C_2$  to rotate relative to  $S$ . To account for the actions of the motors, let  $S$  exert contact forces on  $C_1$  and  $C_2$ , and replace the set of all such forces acting on  $C_i$  with a couple of torque  $\mathbf{M}_i$

together with a force  $\mathbf{K}_i$  applied at the center of  $C_i$ , with  $\mathbf{M}_i$  and  $\mathbf{K}_i$  ( $i = 1, 2$ ) given by

$$\mathbf{M}_1 = \alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2 + \alpha_3 \mathbf{n}_3$$

$$\mathbf{M}_2 = \beta_1 \mathbf{n}_1 + \beta_2 \mathbf{n}_2 + \beta_3 \mathbf{n}_3$$

$$\mathbf{K}_1 = \gamma_1 \mathbf{n}_1 + \gamma_2 \mathbf{n}_2 + \gamma_3 \mathbf{n}_3$$

$$\mathbf{K}_2 = \delta_1 \mathbf{n}_1 + \delta_2 \mathbf{n}_2 + \delta_3 \mathbf{n}_3$$

Determine the generalized active forces  $\tilde{F}_1$ ,  $\tilde{F}_2$ , and  $\tilde{F}_3$  in  $A$  for the system formed by  $S$ ,  $C_1$ , and  $C_2$ . (Keep the law of action and reaction in mind.)

**Results**  $\tilde{F}_1 = -(\alpha_3 + \beta_3)/R$      $\tilde{F}_2 = L(\alpha_3 - \beta_3)/R$      $\tilde{F}_3 = -(\alpha_3 + \beta_3)$

**8.5** When the motors considered in Problem 8.4 are used to cause  $C_1$  and  $C_2$  to rotate in such a way that

$$s\omega^{C_1} = \Omega_1 \mathbf{n}_3 \quad s\omega^{C_2} = \Omega_2 \mathbf{n}_3$$

where  $\Omega_1$  and  $\Omega_2$  are prescribed functions of the time  $t$ , then the system formed by  $S$ ,  $C_1$ , and  $C_2$  possesses one degree of freedom in  $A$  if  $C_1$  and  $C_2$  roll on  $P$ , as was pointed out in Problem 4.12. Show that the associated nonholonomic generalized active force is equal to zero if  $P$  is horizontal.

**8.6** In Fig. P8.6, which deals with the system previously considered in Problems 4.3 and 4.6,  $\mathbf{a}_x$  points vertically downward, while  $\mathbf{a}_y$  and  $\mathbf{a}_z$  are horizontal. Define a generalized speed  $u_1$  as  $u_1 \triangleq \dot{q}_1$ , let  $m_1$  and  $m_2$  be the masses of  $P_1$  and  $P_2$ ,

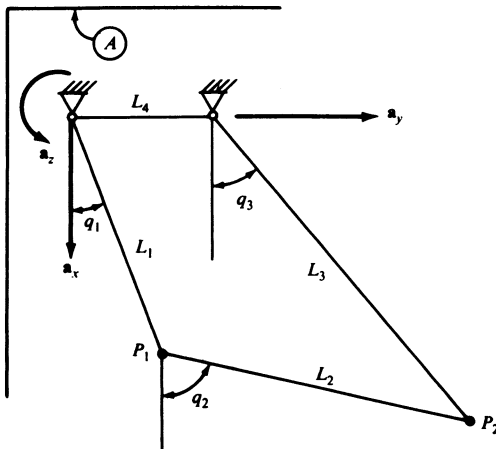


Figure P8.6

respectively, and assume that the links supporting  $P_1$  and  $P_2$  are so light in comparison with  $P_1$  and  $P_2$  that the masses of the links can be neglected. Determine  $(\tilde{F}_1)_y$ , the contribution of gravitational forces to the generalized active force  $\tilde{F}_1$  for the particles  $P_1$  and  $P_2$  in  $A$ , on the basis of the following considerations.

The system  $S$  formed by  $P_1$  and  $P_2$  possesses only one degree of freedom in  $A$ . Hence, a single generalized coordinate, such as  $q_1$ , suffices to characterize the configuration of  $S$  in  $A$ . However, it is convenient to introduce the *pseudo-generalized coordinates*  $q_2$  and  $q_3$  (see Fig. P8.6), which must satisfy the configuration constraint equations

$$L_1 c_1 + L_2 c_2 - L_3 c_3 = 0$$

$$L_1 s_1 + L_2 s_2 - L_3 s_3 - L_4 = 0$$

and to let  $u_2$  and  $u_3$  be *pseudo-generalized speeds* defined as  $u_2 \triangleq \dot{q}_2$  and  $u_3 \triangleq \dot{q}_3$ . Differentiation of the constraint equations with respect to time then permits one to express  $u_2$  and  $u_3$  as in Eqs. (2.13.1), and  $S$  then can be treated as if it were a simple nonholonomic system.

### Result

$$(\tilde{F}_1)_y = -gL_1 \left[ m_1 s_1 + m_2 s_3 \frac{\sin(q_2 - q_1)}{\sin(q_2 - q_3)} \right]$$

**8.7** Figure P8.7 shows 33 pin-connected rods, each of mass  $m$  and length  $L$ , suspended from a horizontal support. Contact forces of magnitudes  $Q$ ,  $R$ , and  $S$  are applied to this system, as indicated in Fig. P8.7.

Using as generalized speeds the quantities  $L\dot{q}_1$ ,  $L\dot{q}_2$ , and  $L\dot{q}_3$ , where  $q_1$ ,  $q_2$ , and  $q_3$  are the angles shown in Fig. P8.7, determine the generalized active forces  $F_1$ ,  $F_2$ , and  $F_3$ .

### Results

$$F_1 = (-Q + R + S)c_1 - 30mgs_1$$

$$F_2 = (Q - R)c_2 - 19mgs_2$$

$$F_3 = Rc_3 - 8mgs_3$$

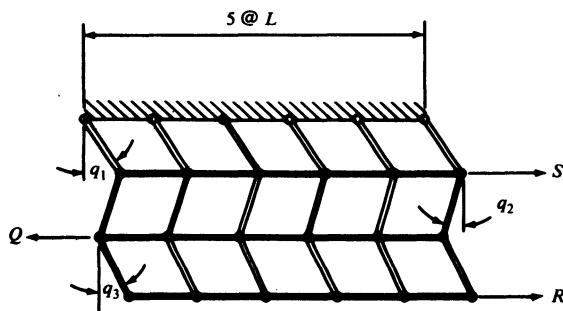


Figure P8.7

**8.8** A rigid block  $B$  of length  $2b$  and mass  $m$  is supported by two elastic beams, each of length  $L$  and flexural rigidity  $EI$ . The beams are “built in” both at their supports and at the points  $P_1$  and  $P_2$  (Fig. P8.8).

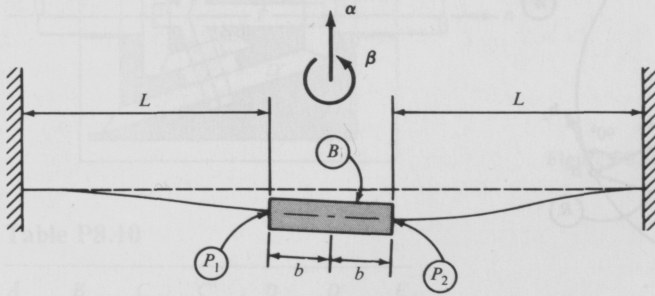


Figure P8.8

Confining attention to planar motions of  $B$  during which  $q_1$  and  $q_2$ , the vertical beam displacements of  $P_1$  and  $P_2$ , respectively, remain small, one can treat  $B$  as a holonomic system possessing two degrees of freedom, and one can replace the set of contact forces exerted on  $B$  by the beam at the left with a couple of torque  $M_1\beta$ , together with a force  $V_1\alpha$  applied at  $P_1$ , where  $\alpha$  and  $\beta$  are unit vectors directed as shown, and  $M_1$  and  $V_1$  are given by

$$M_1 = \frac{12EI}{L^2} \left( \frac{L}{3} \frac{q_2 - q_1}{2b} - \frac{q_1}{2} \right)$$

$$V_1 = \frac{12EI}{L^3} \left( q_1 - \frac{L}{2} \frac{q_2 - q_1}{2b} \right)$$

The set of contact forces exerted on  $B$  by the beam at the right can be replaced similarly.

Taking for generalized speeds,  $u_1$  and  $u_2$ , the time-derivatives of  $q_1$  and  $q_2$ , respectively, determine the generalized active forces  $F_1$  and  $F_2$ .

### Results

$$F_1 = \frac{12EI}{L^3} \left[ - \left( 1 + \frac{L}{2b} + \frac{L^2}{6b^2} \right) q_1 + \frac{L}{2b} \left( 1 + \frac{L}{3b} \right) q_2 \right] + \frac{mg}{2}$$

$$F_2 = \frac{12EI}{L^3} \left[ \frac{L}{2b} \left( 1 + \frac{L}{3b} \right) q_1 - \left( 1 + \frac{L}{2b} + \frac{L^2}{6b^2} \right) q_2 \right] + \frac{mg}{2}$$

**8.9** The ends  $A$  and  $B$  of a uniform rod  $R$  of mass  $m$  are supported by a circular wire  $W$  that lies in a fixed vertical plane and has a radius  $r$ . At the center of  $W$ ,  $A$  and  $B$  subtend an angle  $2\theta$  (Fig. P8.9).

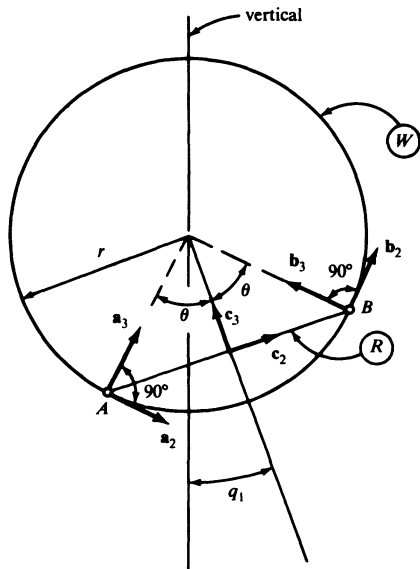


Figure P8.9

(a) Letting  $u_1$  be a generalized speed defined as  $u_1 \triangleq \dot{q}_1$ , where  $q_1$  is the angle shown in Fig. P8.9, and assuming that friction forces exerted on  $R$  by  $W$  at  $A$  and  $B$  are negligible, determine the generalized active force  $F_1$ .

(b) Letting  $-sT_A\mathbf{a}_2$  and  $-sT_B\mathbf{b}_2$  be friction forces exerted on  $R$  by  $W$  at  $A$  and  $B$ , respectively, where  $T_A$  and  $T_B$  are non-negative quantities and  $s \triangleq u_1/|u_1|$ , while  $\mathbf{a}_2$  and  $\mathbf{b}_2$  are unit vectors directed as shown in Fig. P8.9, determine the generalized active force  $F_1$ .

(c) The forces exerted on  $R$  by  $W$  at  $A$  and  $B$  in the radial directions at these points can be expressed as  $N_A\mathbf{a}_3$  and  $N_B\mathbf{b}_3$ , respectively. To bring these into evidence in expressions for generalized active forces, let  $C$  be a reference frame in which the unit vectors  $\mathbf{c}_2$  and  $\mathbf{c}_3$  shown in Fig. P8.9 are fixed, introduce generalized speeds  $u_2$  and  $u_3$  such that the velocity of the center of  $R$  in  $C$  is given by  $u_2\mathbf{c}_2 + u_3\mathbf{c}_3$ , and determine the generalized active forces  $F_1$ ,  $F_2$ , and  $F_3$ .

### Results

- (a)  $F_1 = -mgr \cos \theta \sin q_1$   
 (b)  $F_1 = -r[s(T_A + T_B) + mg \cos \theta \sin q_1]$   
 (c)  $F_1 = -r[s(T_A + T_B) + mg \cos \theta \sin q_1]$   
 $F_2 = -s(T_A + T_B) \cos \theta + (N_A - N_B) \sin \theta - mg \sin q_1$   
 $F_3 = s(T_A - T_B) \sin \theta + (N_A + N_B) \cos \theta - mg \cos q_1$

**8.10** Figure P8.10 is a schematic representation of a reduction gear consisting of a fixed bevel gear  $A$ , moving bevel gears  $B$ ,  $C$ ,  $C'$ ,  $D$ ,  $D'$ , and  $E$ , and an arm  $F$ .  $C$  and  $D$  are rigidly connected to each other, as are  $C'$  and  $D'$ . The number of teeth of each gear is shown in Table P8.10.

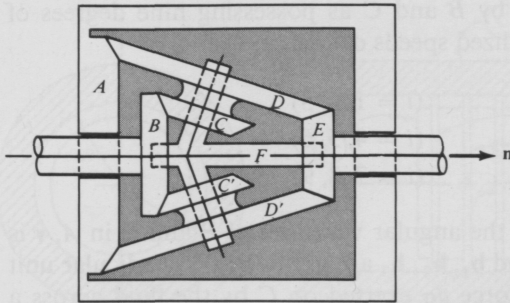


Figure P8.10

Table P8.10

A	B	C	C'	D	D'	E
60	30	30	30	61	61	20

Couples of torques  $T_B \mathbf{n}$  and  $T_E \mathbf{n}$ , where  $\mathbf{n}$  is a unit vector directed as shown in Fig. P8.10, are applied to the shafts carrying  $B$  and  $E$ , respectively. Letting  $u_1$  be a generalized speed defined as  $u_1 \triangleq {}^A \boldsymbol{\omega}^B \cdot \mathbf{n}$ , and assuming that rolling takes place at all points of contact between gears, determine the generalized active force  $F_1$ .

**Result**  $F_1 = T_B + 244T_E$

**8.11** A uniform sphere  $C$  of radius  $R$  is placed in a spherical cavity of a rigid body  $B$ , and  $B$  is free to move in a reference frame  $A$ , as indicated in Fig. P8.11. The radius of the cavity exceeds  $R$  only slightly, and the space between  $B$  and  $C$  is filled with a viscous fluid.

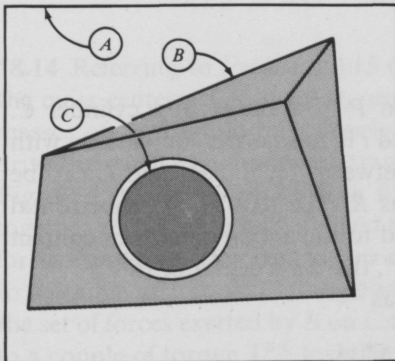


Figure P8.11

Regarding the system  $S$  formed by  $B$  and  $C$  as possessing nine degrees of freedom in  $A$ , let  $u_1, \dots, u_9$  be generalized speeds defined as

$$u_i \triangleq \begin{cases} {}^A\omega^B \cdot \mathbf{b}_i & (i = 1, 2, 3) \\ {}^A\omega^C \cdot \mathbf{b}_{i-3} & (i = 4, 5, 6) \\ \mathbf{v} \cdot \mathbf{b}_{i-6} & (i = 7, 8, 9) \end{cases}$$

where  ${}^A\omega^B$  and  ${}^A\omega^C$  are, respectively, the angular velocities of  $B$  and  $C$  in  $A$ ,  $\mathbf{v}$  is the velocity of the center of  $C$  in  $A$ , and  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are mutually perpendicular unit vectors fixed in  $B$ . Assume that the force  $d\sigma$  exerted on  $C$  by the fluid across a differential element  $d\Sigma$  of the surface  $C$  is given by

$$d\sigma = -c {}^B\mathbf{v}^P da$$

where  $c$  is a constant,  ${}^B\mathbf{v}^P$  is the velocity in  $B$  of any point  $P$  of  $C$  lying within  $d\Sigma$ , and  $da$  is the area of  $d\Sigma$ ; further, that at  $P$  the force exerted on  $B$  by the fluid across a differential element of the surface of the cavity is equal to  $-d\sigma$ .

Letting  $F_1, \dots, F_9$  be the generalized active forces for  $S$  in  $A$ , determine the contributions to  $F_2$  and  $F_4$  of forces exerted on  $B$  and  $C$  by the fluid.

**Results**  $\frac{8}{3}\pi cR^4(u_5 - u_2); \quad \frac{8}{3}\pi cR^4(u_1 - u_4)$

**8.12** Referring to the example in Sec. 4.8, suppose that  $Q$  slides on  $P$  subject to the laws of Coulomb friction, and let  $\mu'$  be the coefficient of kinetic friction for  $A$  and  $P$ . Under these circumstances, the quantities  $Q_1, Q_2$ , and  $Q_3$  appearing in Eq. (4.8.14) can be expressed as functions of  $\mu', R, M, g, e, f, \theta, u_1, u_2$ , and a nonholonomic generalized active force  $\tilde{F}_3$  associated with a generalized speed  $u_3$  introduced for the purpose of bringing  $Q_1$  into evidence. Taking  $u_3 \triangleq {}^F\omega^A \cdot \mathbf{a}_3$ , determine these functions.

**Results**

$$\begin{aligned} Q_1 &= (eMg \cos \theta - \tilde{F}_3) / [f - \mu' Ru_2 / (u_2^2 + f^2 u_1^2)^{1/2}] \\ Q_2 &= -\mu' u_2 Q_1 / (u_2^2 + f^2 u_1^2)^{1/2} \\ Q_3 &= -\mu' u_1 f Q_1 / (u_2^2 + f^2 u_1^2)^{1/2} \end{aligned}$$

**8.13** Figure P8.13 shows two views of a piston  $P$  of mass  $M$  in a cylinder  $C$ .  $R_1, R_2$ , and  $R_3$  are piston rings of thickness  $t$ , and  $t$  is so small in comparison with  $R$ , the internal radius of  $C$ , that the contacts between  $R_1, R_2, R_3$  and  $C$  can be regarded as taking place along circles of radius  $R$ . The axis of  $C$  is horizontal and parallel to a unit vector  $\mathbf{i}$ , and  $P$  is subjected to the action of a set of contact forces equivalent to a force  $E\mathbf{i}$  applied to  $P$  at  $P^*$ , the mass center of  $P$ .

Two generalized speeds,  $u_1$  and  $u_2$ , defined as

$$u_1 \triangleq \omega \cdot \mathbf{i} \quad u_2 \triangleq \mathbf{v}^* \cdot \mathbf{i}$$



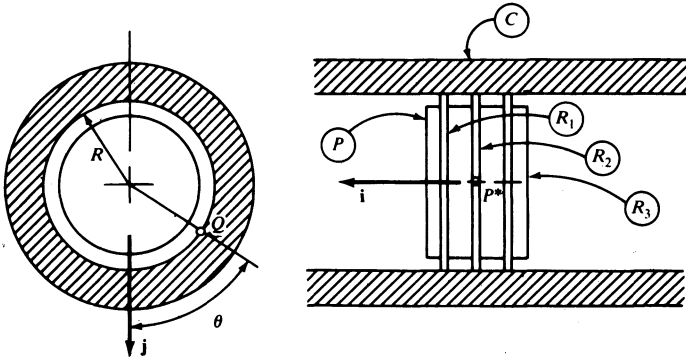


Figure P8.13

where  $\omega$  is the angular velocity of  $P$  in  $C$  and  $\mathbf{v}^*$  is the velocity of  $P^*$  in  $C$ , characterize motions of  $P$  in  $C$  if  $P$  is free to rotate and translate. A third generalized speed,  $u_3$ , defined as

$$u_3 \triangleq \mathbf{v}^* \cdot \mathbf{j}$$

where  $\mathbf{j}$  is a unit vector directed vertically downward, can be introduced to bring into evidence radial contact forces exerted on  $R_1$ ,  $R_2$ , and  $R_3$  by  $C$ .

Assuming that  $n$ , the contact pressure (force per unit of length) at a point  $Q$  situated on any one of  $R_1$ ,  $R_2$ , or  $R_3$  as shown in Fig. P8.13, is given by

$$n = \alpha + \beta \cos\left(\frac{\theta}{2}\right) \quad (-\pi \leq \theta \leq \pi)$$

where  $\alpha$  and  $\beta$  are constants, and letting  $\mu'$  be the coefficient of kinetic friction for  $C$  and  $R_1$ ,  $R_2$ ,  $R_3$ , determine the generalized active forces  $F_1$ ,  $F_2$ , and  $F_3$  for motions during which  $u_3 = 0$ .

### Results

$$\begin{aligned} F_1 &= -6\mu'R^3(\pi\alpha + 2\beta)u_1[(Ru_1)^2 + u_2^2]^{-1/2} \\ F_2 &= -6\mu'R(\pi\alpha + 2\beta)u_2[(Ru_1)^2 + u_2^2]^{-1/2} + E \\ F_3 &= -4R\beta + Mg \end{aligned}$$

**\*8.14** Referring to Problem 3.15 (see also Problem 4.17), let  $A^*$ ,  $B^*$ ,  $C^*$ ,  $D^*$  be the mass centers of  $A$ ,  $B$ ,  $C$ ,  $D$ , respectively, and designate as  $m_A$ ,  $m_B$ ,  $m_C$ ,  $m_D$  the masses of  $A$ ,  $B$ ,  $C$ ,  $D$ , respectively. The pinion gears have negligible masses. To drive the robot arm, forces are transmitted from  $E$  to  $A$  via a motor (not shown in Fig. P3.15). The set of all such forces is equivalent to a couple of torque  $\mathbf{T}^{E/A}$  together with a force  $\mathbf{K}^{E/A}$  applied to  $A$  at a point on the axis of the hub. The set of forces exerted by  $A$  on  $B$  by means of the motor connecting  $A$  to  $B$  at  $P$  is equivalent to a couple of torque  $\mathbf{T}^{A/B}$  together with a force  $\mathbf{K}^{A/B}$  applied to  $B$  at  $P$ . Finally, the set of forces exerted by  $B$  on  $C$  through the rack-and-pinion drive is equivalent to a couple of torque  $\mathbf{T}^{B/C}$  together with a force  $\mathbf{K}^{B/C}$  applied to  $C$  at  $C^*$ .

Expressing  $\mathbf{T}^{E/A}$ ,  $\mathbf{K}^{E/A}$ ,  $\mathbf{T}^{A/B}$ ,  $\mathbf{K}^{A/B}$ ,  $\mathbf{T}^{B/C}$ , and  $\mathbf{K}^{B/C}$  as, respectively,

$$\begin{aligned}\mathbf{T}^{E/A} &= T_1^{E/A} \mathbf{a}_1 + T_2^{E/A} \mathbf{a}_2 + T_3^{E/A} \mathbf{a}_3 \\ \mathbf{K}^{E/A} &= K_1^{E/A} \mathbf{a}_1 + K_2^{E/A} \mathbf{a}_2 + K_3^{E/A} \mathbf{a}_3 \\ \mathbf{T}^{A/B} &= T_1^{A/B} \mathbf{b}_1 + T_2^{A/B} \mathbf{b}_2 + T_3^{A/B} \mathbf{b}_3 \\ \mathbf{K}^{A/B} &= K_1^{A/B} \mathbf{b}_1 + K_2^{A/B} \mathbf{b}_2 + K_3^{A/B} \mathbf{b}_3 \\ \mathbf{T}^{B/C} &= T_1^{B/C} \mathbf{b}_1 + T_2^{B/C} \mathbf{b}_2 + T_3^{B/C} \mathbf{b}_3 \\ \mathbf{K}^{B/C} &= K_1^{B/C} \mathbf{b}_1 + K_2^{B/C} \mathbf{b}_2 + K_3^{B/C} \mathbf{b}_3\end{aligned}$$

and letting  $g$  be the local gravitational acceleration, determine the generalized active forces  $F_1$ ,  $F_2$ , and  $F_3$  for the robot arm in  $E$ , where  $F_i$  is associated with the generalized speed  $u_i$  ( $i = 1, 2, 3$ ) defined in Problem 3.15.

### Results

$$\begin{aligned}F_1 &= T_1^{E/A} \\ F_2 &= T_2^{A/B} - g[(m_B L_B + m_C Z_9 + m_D Z_{14})c_1 - m_D p_1 s_1] \\ F_3 &= K_3^{B/C} - g(m_C + m_D)s_1\end{aligned}$$

**8.15** Letting  $S$  be the set of two particles  $P_1$  and  $P_2$  considered in Problem 8.1, form expressions for the generalized inertia forces  $F_1^*$ ,  $F_2^*$ ,  $F_3^*$  for  $S$  in  $A$ , doing so for each of the three sets of generalized speeds defined in Problem 8.1. Comment briefly on the relative merits of the three sets of generalized speeds.

### Results

$$\begin{aligned}F_1^* &= \{-(m_1 + m_2)(\dot{u}_1 \sec \omega t + 2\omega \dot{q}_1 \tan \omega t) \\ &\quad + m_2[(\omega^2 + u_3^2)c_3 + \dot{u}_3 s_3]\} \sec \omega t \\ F_2^* &= -(m_1 + m_2)\dot{u}_2 + Lm_2(u_3^2 s_3 - \dot{u}_3 c_3) \\ F_3^* &= -m_2 L[L(\dot{u}_3 + \omega^2 s_3 c_3) - (\dot{u}_1 \sec \omega t + 2\omega \dot{q}_1 \tan \omega t)s_3 + \dot{u}_2 c_3] \\ F_1^* &= -(m_1 + m_2)(\dot{u}_1 - \omega^2 q_1 c_3 - u_2 u_3) + Lm_2(\omega^2 c_3^2 + u_3^2) \\ F_2^* &= -(m_1 + m_2)(\dot{u}_2 + u_3 u_1 + \omega^2 q_1 s_3) - Lm_2(\dot{u}_3 + \omega^2 s_3 c_3) \\ F_3^* &= -m_2 L[\dot{u}_2 + u_3 u_1 + \omega^2 q_1 s_3 + L(\dot{u}_3 + \omega^2 s_3 c_3)] \\ F_1^* &= -(m_1 + m_2)(\dot{u}_1 - \omega^2 q_1) + Lm_2(\omega^2 c_3 + \dot{u}_3 s_3 + u_3^2 c_3) \\ F_2^* &= -(m_1 + m_2)\dot{u}_2 + Lm_2(u_3^2 s_3 - \dot{u}_3 c_3) \\ F_3^* &= -m_2 L[L(\dot{u}_3 + \omega^2 s_3 c_3) - (\dot{u}_1 - \omega^2 q_1)s_3 + \dot{u}_2 c_3]\end{aligned}$$

**8.16** Making the same assumptions as in Problem 8.2, use Eqs. (4.11.1) to determine the generalized inertia forces  $\tilde{F}_1^*$  and  $\tilde{F}_2^*$  for  $S$  in  $A$ . Check the results by using Eqs. (4.11.4) in conjunction with results from Problem 8.15, and comment briefly on the relative merits of Eqs. (4.11.1) and Eqs. (4.11.4).

**Results**

$$\begin{aligned}\tilde{F}_1^* &= (m_1 + m_2)(\omega^2 q_1 c_3 - \dot{u}_1) - m_1 u_2^2/L + m_2 L \omega^2 c_3^2 \\ \tilde{F}_2^* &= -m_1(\dot{u}_2 + \omega^2 q_1 s_3 - u_1 u_2/L)\end{aligned}$$

**8.17** The system  $S$  considered in the example in Sec. 4.8 and shown in Fig. 4.8.1 has the following inertia properties:  $A$  has a mass  $m_A$  and moment of inertia  $I_A$  about a line parallel to  $\mathbf{a}_1$  and passing through the mass center  $A^*$  of  $A$ , which is situated on line  $DE$  at a distance  $a$  from point  $D$ . The wheels  $B$  and  $C$  each have a mass  $m_B$  and moment of inertia  $J$  about the line joining the wheel centers (which are the wheels' mass centers and are separated by a distance  $2b$ ), and each wheel has a moment of inertia  $K$  about any line that passes through the center of the wheel and is perpendicular to the line joining the wheel centers.

Defining generalized speeds  $u_1$  and  $u_2$  as in Eqs. (4.8.9), and  $u_3$  as in Problem 8.12, determine the generalized inertia forces  $\tilde{F}_1^*$ ,  $\tilde{F}_2^*$ , and  $\tilde{F}_3^*$  for  $S$  in  $F$ .

**Results**

$$\begin{aligned}\tilde{F}_1^* &= -(I_A + 2Jb^2/R^2 + 2K + m_A a^2 + 2m_B b^2)\dot{u}_1 - m_A a u_1 u_2 \\ \tilde{F}_2^* &= -(m_A + 2m_B + 2J/R^2)\dot{u}_2 + m_A a u_1^2 \\ \tilde{F}_3^* &= 0\end{aligned}$$

**\*8.18** The equations that follow furnish useful expressions for  $\mathbf{T}^*$ , the inertia torque for a rigid body  $B$  in a reference frame  $A$ . Establish the validity of each of these equations.

(a) When  $\boldsymbol{\omega}$ , the angular velocity of  $B$  in  $A$ , can be expressed as  $\boldsymbol{\omega} = \omega \mathbf{n}_a$ , where  $\mathbf{n}_a$  is a unit vector fixed in  $A$ , then

$$\mathbf{T}^* = -\dot{\omega} I_a \mathbf{n}_a + (\omega^2 I_{ca} - \dot{\omega} I_{ab}) \mathbf{n}_b - (\omega^2 I_{ab} + \dot{\omega} I_{ca}) \mathbf{n}_c$$

where  $\mathbf{n}_b$  and  $\mathbf{n}_c$  are unit vectors perpendicular to  $\mathbf{n}_a$  and to each other, and are oriented such that  $\mathbf{n}_a = \mathbf{n}_b \times \mathbf{n}_c$ ;  $I_a$  is the central moment of inertia of  $B$  with respect to a line parallel to  $\mathbf{n}_a$ ;  $I_{ab}$  is the central product of inertia of  $B$  for  $\mathbf{n}_a$  and  $\mathbf{n}_b$ ; and  $I_{ca}$  is the central product of inertia of  $B$  for  $\mathbf{n}_c$  and  $\mathbf{n}_a$ .

(b) If  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a right-handed set of mutually perpendicular unit vectors fixed in  $B$ , and  $\omega_i \triangleq \boldsymbol{\omega} \cdot \mathbf{b}_i$  ( $i = 1, 2, 3$ ), where  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $A$ , then

$$\begin{aligned}\mathbf{T}^* &= -[I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{13}\dot{\omega}_3 + (I_{31}\omega_2 - I_{12}\omega_3)\omega_1 \\ &\quad + I_{23}(\omega_2^2 - \omega_3^2) + (I_{33} - I_{22})\omega_2\omega_3]\mathbf{b}_1 \\ &\quad -[I_{22}\dot{\omega}_2 + I_{23}\dot{\omega}_3 + I_{21}\dot{\omega}_1 + (I_{12}\omega_3 - I_{23}\omega_1)\omega_2 \\ &\quad + I_{31}(\omega_3^2 - \omega_1^2) + (I_{11} - I_{33})\omega_3\omega_1]\mathbf{b}_2 \\ &\quad -[I_{33}\dot{\omega}_3 + I_{31}\dot{\omega}_1 + I_{32}\dot{\omega}_2 + (I_{23}\omega_1 - I_{31}\omega_2)\omega_3 \\ &\quad + I_{12}(\omega_1^2 - \omega_2^2) + (I_{22} - I_{11})\omega_1\omega_2]\mathbf{b}_3\end{aligned}$$

where  $I_{jk}$  is the central inertia scalar for  $\mathbf{b}_j$  and  $\mathbf{b}_k$  ( $j, k = 1, 2, 3$ ).

(c) If  $\mathbf{H}$  is the central angular momentum of  $B$  in  $A$ , then

$$\mathbf{T}^* = - \frac{{}^A d\mathbf{H}}{dt}$$

**\*8.19**  $(\tilde{F}_r^*)_B$ , the contribution of a rigid body  $B$  to the generalized inertia force  $\tilde{F}_r^*$  ( $r = 1, \dots, p$ ) in a reference frame  $A$ , can be expressed as

$$(\tilde{F}_r^*)_B = \tilde{\omega}_r \cdot \mathbf{T}^Q + \tilde{\mathbf{v}}_r^Q \cdot \mathbf{R}^* \quad (r = 1, \dots, p) \quad (a)$$

where  $\tilde{\omega}_r$  and  $\mathbf{R}^*$  are defined as in connection with Eqs. (4.11.7), but  $\mathbf{T}^Q$  and  $\tilde{\mathbf{v}}_r^Q$  differ from  $\mathbf{T}^*$  and  $\tilde{\mathbf{v}}_r^*$ , respectively, being defined as follows: Let  $Q$  be a point of  $B$ ,  $\mathbf{r}^{QP_i}$  the position vector from  $Q$  to a generic particle  $P_i$  of  $B$ ,  $m_i$  the mass of  $P_i$ , and  $\mathbf{a}_i$  the acceleration of  $P_i$  ( $i = 1, \dots, \beta$ ) in reference frame  $A$ . Then

$$\mathbf{T}^Q \triangleq - \sum_{i=1}^{\beta} m_i \mathbf{r}^{QP_i} \times \mathbf{a}_i \quad (b)$$

while  $\tilde{\mathbf{v}}_r^Q$  is the  $r$ th nonholonomic partial velocity of  $Q$  in  $A$ . Furthermore,  $\mathbf{T}^Q$  [compare with  $\mathbf{T}^*$  as given in Eq. (4.11.8)] can be expressed as

$$\mathbf{T}^Q = -m\mathbf{r}^{QB^*} \times \mathbf{a}^Q - \boldsymbol{\alpha} \cdot \mathbf{I}^Q - \boldsymbol{\omega} \times \mathbf{I}^Q \cdot \boldsymbol{\omega} \quad (c)$$

where  $m$  is the mass of  $B$ ,  $\mathbf{r}^{QB^*}$  is the position vector from  $Q$  to  $B^*$ ,  $\mathbf{a}^Q$  is the acceleration of  $Q$  in  $A$ ,  $\boldsymbol{\alpha}$  and  $\boldsymbol{\omega}$  are, respectively, the angular acceleration of  $B$  in  $A$  and the angular velocity of  $B$  in  $A$ , and  $\mathbf{I}^Q$  is the inertia dyadic of  $B$  relative to  $Q$ .

Establish the validity of Eqs. (a) and (c), and devise an example to illustrate the utility of these relationships.

**\*8.20** Referring to Problem 3.15 (see also Problems 4.17 and 8.14), let  $A$ ,  $B$ ,  $C$ , and  $D$  have the following mass distribution properties. The central principal axes of  $A$  are parallel to  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  (see Fig. P3.15), and the associated moments of inertia have the values  $A_1, A_2, A_3$ , respectively. The central principal axes of  $B$  and  $C$  are parallel to  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  (see Fig. P3.15), and the associated moments of inertia have the values  $B_1, B_2, B_3$  and  $C_1, C_2, C_3$ , respectively. Central inertia scalars  $D_{ij}$  ( $i, j = 1, 2, 3$ ) for  $D$  are defined as

$$D_{ij} \triangleq \mathbf{b}_i \cdot \mathbf{I}^D \cdot \mathbf{b}_j \quad (i, j = 1, 2, 3)$$

where  $\mathbf{I}^D$  is the central inertia dyadic of  $D$ .

Letting  $\mathbf{T}_A^*, \mathbf{T}_B^*, \mathbf{T}_C^*$ , and  $\mathbf{T}_D^*$  denote the inertia torques for  $A$ ,  $B$ ,  $C$ , and  $D$  in  $E$ , express  $\mathbf{T}_A^*$  in terms of  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , and  $\mathbf{T}_B^*, \mathbf{T}_C^*, \mathbf{T}_D^*$  in terms of  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ .

To facilitate the writing of results, introduce  $k_1, \dots, k_{16}$  and  $Z_{35}, \dots, Z_{54}$  as

$$\begin{aligned} k_1 &\triangleq B_2 - B_3 & k_2 &\triangleq B_3 - B_1 & k_3 &\triangleq B_1 - B_2 & k_4 &\triangleq C_2 - C_3 \\ k_5 &\triangleq C_3 - C_1 & k_6 &\triangleq C_1 - C_2 & k_7 &\triangleq D_{33} - D_{22} & k_8 &\triangleq D_{11} - D_{33} \\ k_9 &\triangleq D_{22} - D_{11} & k_{10} &\triangleq B_1 + k_1 & k_{11} &\triangleq B_3 - k_3 & k_{12} &\triangleq C_1 + k_4 \\ k_{13} &\triangleq C_3 - k_6 & k_{14} &\triangleq D_{11} - k_7 & k_{15} &\triangleq D_{31} + D_{13} & k_{16} &\triangleq D_{33} + k_9 \end{aligned}$$

$$\begin{aligned} Z_{35} &\triangleq c_1 B_1 & Z_{36} &\triangleq Z_3 k_{10} & Z_{37} &\triangleq Z_2 Z_1 & Z_{38} &\triangleq -Z_{37} k_2 \\ Z_{39} &\triangleq s_1 B_3 & Z_{40} &\triangleq Z_4 k_{11} & Z_{41} &\triangleq c_1 C_1 & Z_{42} &\triangleq Z_3 k_{12} \\ Z_{43} &\triangleq -Z_{37} k_5 & Z_{44} &\triangleq s_1 C_3 & Z_{45} &\triangleq Z_4 k_{13} & Z_{46} &\triangleq Z_1^2 \\ Z_{47} &\triangleq u_2^2 & Z_{48} &\triangleq Z_2^2 & Z_{49} &\triangleq D_{11} c_1 + D_{13} s_1 \\ Z_{50} &\triangleq k_{14} Z_3 + k_{15} Z_4 - D_{12} Z_{37} + D_{23} (Z_{47} - Z_{48}) & Z_{51} &\triangleq D_{23} s_1 + D_{21} c_1 \\ Z_{52} &\triangleq D_{31} (Z_{48} - Z_{46}) + k_8 Z_{37} & Z_{53} &\triangleq D_{33} s_1 + D_{31} c_1 \\ Z_{54} &\triangleq k_{15} Z_3 + k_{16} Z_4 + D_{23} Z_{37} + D_{12} (Z_{46} - Z_{47}) \end{aligned}$$

### Results

$$\begin{aligned} \mathbf{T}_A^* &= -\dot{u}_1 A_1 \mathbf{a}_1 \\ \mathbf{T}_B^* &= -(\dot{u}_1 Z_{35} + Z_{36}) \mathbf{b}_1 - (\dot{u}_2 B_2 + Z_{38}) \mathbf{b}_2 - (\dot{u}_1 Z_{39} + Z_{40}) \mathbf{b}_3 \\ \mathbf{T}_C^* &= -(\dot{u}_1 Z_{41} + Z_{42}) \mathbf{b}_1 - (\dot{u}_2 C_2 + Z_{43}) \mathbf{b}_2 - (\dot{u}_1 Z_{44} + Z_{45}) \mathbf{b}_3 \\ \mathbf{T}_D^* &= -(\dot{u}_1 Z_{49} + \dot{u}_2 D_{12} + Z_{50}) \mathbf{b}_1 - (\dot{u}_1 Z_{51} + \dot{u}_2 D_{22} + Z_{52}) \mathbf{b}_2 \\ &\quad - (\dot{u}_1 Z_{53} + \dot{u}_2 D_{32} + Z_{54}) \mathbf{b}_3 \end{aligned}$$

**\*8.21** Referring to Problem 3.15 (see also Problem 4.17, 8.14, and 8.20), determine the generalized inertia forces  $F_1^*$ ,  $F_2^*$ , and  $F_3^*$  for the robot arm in  $E$ , where  $F_i^*$  is associated with the generalized speed  $u_i$  ( $i = 1, 2, 3$ ).

### Results

$$\begin{aligned} F_1^* &= -\dot{u}_1 [A_1 + c_1 (Z_{35} + Z_{41} + Z_{49}) + s_1 (Z_{39} + Z_{44} + Z_{53}) \\ &\quad + m_A L_A^2 + m_B Z_6^2 + m_C Z_{10}^2 + m_D (Z_{13}^2 + Z_{15}^2 + Z_{16}^2)] \\ &\quad - \dot{u}_2 [c_1 D_{12} + s_1 D_{32} + m_D (Z_{13} Z_{14} - Z_{16} p_1)] - \dot{u}_3 m_D Z_{16} \\ &\quad - [c_1 (Z_{36} + Z_{42} + Z_{50}) + s_1 (Z_{40} + Z_{45} + Z_{54}) + m_B Z_6 Z_{23} \\ &\quad + m_C Z_{10} Z_{27} + m_D (Z_{13} Z_{32} + Z_{15} Z_{33} + Z_{16} Z_{34})] \\ F_2^* &= -\dot{u}_1 [Z_{51} + m_D (Z_{14} Z_{13} - p_1 Z_{16})] - \dot{u}_2 [B_2 + C_2 + D_{22} + m_B L_B^2 \\ &\quad + m_C Z_9^2 + m_D (Z_{14}^2 + p_1^2)] + \dot{u}_3 m_D p_1 - [Z_{38} + Z_{43} + Z_{52} \\ &\quad + m_B L_B Z_{22} + m_C Z_9 Z_{26} + m_D (Z_{14} Z_{32} - p_1 Z_{34})] \\ F_3^* &= -\dot{u}_1 m_D Z_{16} + \dot{u}_2 m_D p_1 - \dot{u}_3 (m_C + m_D) - (m_C Z_{28} + m_D Z_{34}) \end{aligned}$$

**\*8.22** Letting  $S$  be a simple nonholonomic system possessing generalized coordinates  $q_1, \dots, q_n$  and generalized speeds  $u_1, \dots, u_p$  in a reference frame  $A$ , and letting  $G$  be the Gibbs function for  $S$  in  $A$  (see Problem 6.16), show that the generalized inertia forces  $\tilde{F}_1^*, \dots, \tilde{F}_p^*$  for  $S$  in  $A$  can be expressed as

$$\tilde{F}_r^* = - \frac{\partial G}{\partial \dot{u}_r} \quad (r = 1, \dots, p)$$

if  $G$  is regarded as a function of  $q_1, \dots, q_n, u_1, \dots, u_p$ , and  $\dot{u}_1, \dots, \dot{u}_p$ . [The formula developed in Problem 6.16 can be used in conjunction with the present result to find the contribution of a rigid body  $B$  to  $\tilde{F}_r^*$  ( $r = 1, \dots, p$ ). When using the formula for this purpose, one can omit the term  $\omega^2 \omega \cdot \mathbf{I} \cdot \omega$ , for this term does not contain any of  $\dot{u}_1, \dots, \dot{u}_p$ , and hence cannot contribute to  $\partial G / \partial \dot{u}_r$  ( $r = 1, \dots, p$ ).]

## PROBLEM SET 9

(Secs. 5.1–5.3)

**9.1** A system  $S$  of two particles  $P_1$  and  $P_2$  of masses  $m_1$  and  $m_2$  moves in such a way that the distance  $r$  between  $P_1$  and  $P_2$  is free to vary with time. Assuming that no forces act on  $P_1$  and  $P_2$  other than the gravitational forces exerted by the particles on each other, show that  $-Gm_1m_2/r + C$ , where  $G$  is the universal gravitational constant and  $C$  is any function of time, is a potential energy of  $S$ .

**9.2** A uniform thin rod  $R$  of cross-sectional area  $A$  is partially immersed in a fluid of mass density  $\rho$ .  $R$  can move in such a way that  $q_1$ , the distance from the immersed end of  $R$  to the surface of the fluid, and  $q_2$ , the angle between the axis of  $R$  and the local vertical, are free to vary. Letting  $\beta$  be the set of buoyancy forces exerted on  $R$  by the fluid, show that  $g\rho A(q_1^2/2) \sec q_2$  is a potential energy contribution of  $\beta$  for  $R$ .

**\*9.3** Referring to Problem 2.7, suppose that  $C$  is of uniform density and has a mass  $m$ , and let  $\gamma$  be the set of gravitational forces acting on  $C$ . Verify that  $mgR \cos q_2$  is a potential energy contribution of  $\gamma$  for  $C$ , and show that this function is a potential energy of  $C$  in  $A$  when  $C$  is rolling on  $H$ , but not when slipping is taking place at  $P$ , unless the contact between  $H$  and  $P$  is frictionless. (Expressions for generalized active forces are available in Problem 8.3.)

**9.4** Show that  $-g(m_1L_1c_1 + m_2L_3c_3)$  is a potential energy of the system of two particles considered in Problem 8.6. Verify that the expression for  $(\tilde{F}_1)_\gamma$  found in Problem 8.6 can be obtained by replacing  $\tilde{F}_r$  with  $(\tilde{F}_r)_\gamma$ , and  $V$  with  $-g(m_1L_1c_1 + m_2L_3c_3)$ , in Eqs. (5.1.14).

**9.5** Referring to Problem 8.7, determine with the aid of Eq. (5.2.2) a potential energy contribution  $V_\gamma$  of the gravitational forces acting on the 33 bars; use Eqs. (5.1.2) with  $V$  replaced with  $V_\gamma$  and  $F_r$  replaced with  $(F_r)_\gamma$  to find the contributions  $(F_r)_\gamma$  of the gravitational forces to the generalized active forces  $F_r$  ( $r = 1, 2, 3$ ). Comment briefly on the relative merits of the method used to obtain these contributions in Problem 8.7, on the one hand, and the method employed in the present problem, on the other hand.

**9.6** Construct a potential energy  $V$  of the system described in Problem 8.8.

**Result**

$$V = \frac{6EI}{L^3} \left[ \left( 1 + \frac{L}{2b} + \frac{L^2}{6b^2} \right) (q_1^2 + q_2^2) - \frac{L}{b} \left( 1 + \frac{L}{3b} \right) q_1 q_2 \right] - \frac{mg}{2} (q_1 + q_2)$$

**9.7** Referring to Problem 8.9, suppose that the plane containing  $W$  is made to rotate with a prescribed angular speed about a vertical line passing through the center of  $W$ . Show that  $-mgr \cos \theta \cos q_1$  is a potential energy of  $R$  so long as the contacts between  $R$  and  $W$  at  $A$  and  $B$  are perfectly smooth.

**\*9.8** Referring to the example in Sec. 4.8 (see Fig. 4.8.1), let  $q_1$  be the angle between  $\mathbf{n}_2$  and  $\mathbf{a}_2$  (see Fig. 4.8.2) as before; define  $q_2$  and  $q_3$  as

$$q_2 \triangleq \mathbf{p} \cdot \mathbf{n}_2 \quad q_3 \triangleq \mathbf{p} \cdot \mathbf{n}_3$$

where  $\mathbf{p}$  is the position vector from a point fixed in  $F$  to point  $D$ ; and introduce wheel rotation angles  $q_4$  and  $q_5$  as shown in Fig. P9.8, where  $L_B$  and  $L_C$  are lines fixed in  $B$  and  $C$ , respectively. Assume that no friction forces come into play at

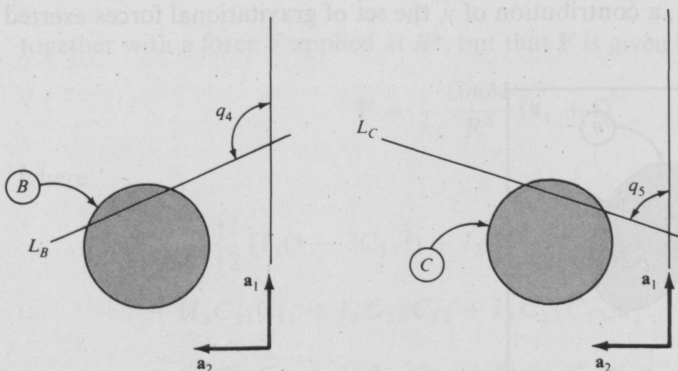


Figure P9.8

point  $Q$  in Fig. 4.8.1, and that  $B$  and  $C$  are driven by motors attached to  $A$  in such a way that the sets of contact forces exerted by  $A$  on  $B$  and by  $A$  on  $C$  are equivalent to couples of torques  $\mathbf{T}_B$  and  $\mathbf{T}_C$ , respectively, with

$$\mathbf{T}_B = T_B \mathbf{a}_3 \quad \mathbf{T}_C = T_C \mathbf{a}_3$$

where  $T_B$  and  $T_C$  are functions of  $q_1, \dots, q_5$ .

Show that  $V(q_1, \dots, q_5, t)$  is a potential energy of  $S$  in  $F$  if

$$\begin{aligned} \frac{\partial V}{\partial q_1} + \frac{b}{R} \left( \frac{\partial V}{\partial q_4} - \frac{\partial V}{\partial q_5} \right) &= \frac{b}{R} (T_C - T_B) - Mge \sin \theta c_1 \\ \frac{\partial V}{\partial q_2} c_1 + \frac{\partial V}{\partial q_3} s_1 + \frac{1}{R} \left( \frac{\partial V}{\partial q_4} + \frac{\partial V}{\partial q_5} \right) &= -\frac{1}{R} (T_C + T_B) - Mg \sin \theta s_1 \end{aligned}$$

**9.9** In Fig. P9.9,  $P$  is a particle fixed in a reference frame  $A$ , and  $B^*$  is the mass center of a rigid body  $B$ ;  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  form a right-handed set of mutually perpendicular unit vectors such that  $\mathbf{a}_1$  points from  $P$  to  $B^*$ ;  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  form a similar set of unit vectors parallel to central principal axes of  $B$ .

Assume throughout what follows that the set  $\gamma$  of all gravitational forces exerted by  $P$  on  $B$  can be replaced with a couple of torque  $\mathbf{T}$  given by

$$\mathbf{T} = \frac{3Gm}{R^3} \mathbf{a}_1 \times \mathbf{I} \cdot \mathbf{a}_1$$

together with a force  $\mathbf{F}$  applied at  $B^*$  and given by

$$\mathbf{F} = -\frac{GmM}{R^2} \mathbf{a}_1$$

where  $G$  is the universal gravitational constant,  $m$  is the mass of  $P$ ,  $R$  is the distance from  $P$  to  $B^*$ ,  $\mathbf{I}$  is the central inertia dyadic of  $B$ , and  $M$  is the mass of  $B$ .

(a) If  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{a}_3$  are fixed in  $A$  and the orientation of  $B$  in  $A$  is specified in terms of three angles  $q_1, q_2, q_3$  like those used in Problem 1.1 to orient  $D$  in  $A$  (see Fig. P1.1), then  $V_\gamma$ , a contribution of  $\gamma$ , the set of gravitational forces exerted

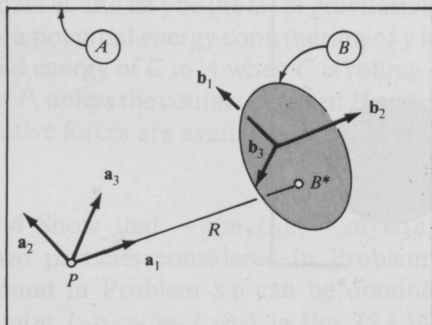


Figure P9.9



by  $P$  on  $B$ , to a potential energy of  $B$  in  $A$ , can be expressed in terms of  $q_1, q_2, q_3$  so long as  $B^*$  is kept fixed in  $A$ . Defining generalized speeds  $u_1, u_2, u_3$  for  $B$  in  $A$  as

$$u_i \triangleq {}^A\boldsymbol{\omega}^B \cdot \mathbf{b}_i \quad (i = 1, 2, 3)$$

and letting  $I_j \triangleq \mathbf{b}_j \cdot \mathbf{I} \cdot \mathbf{b}_j$  ( $j = 1, 2, 3$ ), determine  $V_\gamma$ .

(b) Use the result of part (a) to determine  $(F_1)_\gamma, (F_2)_\gamma, (F_3)_\gamma$ , the contributions of  $\gamma$  to the generalized active forces for  $B$  in  $A$  corresponding to generalized speeds  $u_1, u_2, u_3$  defined as

$$u_i \triangleq \dot{q}_i \quad (i = 1, 2, 3)$$

(c) Assuming that  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{a}_3$  are fixed in  $A$ , but that  $B^*$  can move in  $A$  along line  $PB^*$ , which means that  $B$  has four degrees of freedom in  $A$ , show that  $V_\gamma$  does not exist.

### Results

$$(a) \quad V_\gamma = -\frac{3Gm}{2R^3} [(I_1 - I_3)s_2^2 + (I_1 - I_2)c_2^2s_3^2]$$

$$(b) \quad (F_1)_\gamma = 0$$

$$(F_2)_\gamma = \frac{3Gm}{R^3} s_2c_2(I_1c_3^2 + I_2s_3^2 - I_3)$$

$$(F_3)_\gamma = \frac{3Gm}{R^3} (I_1 - I_2)c_2^2s_3c_3$$

**\*9.10** Referring to Problem 9.9, and assuming that, as in part (c),  $B^*$  is free to move in  $A$  along line  $PB^*$ , suppose that  $\gamma$  is replaced with a couple of torque  $\mathbf{T}$  given once again by

$$\mathbf{T} = \frac{3Gm}{R^3} \mathbf{a}_1 \times \mathbf{I} \cdot \mathbf{a}_1$$

together with a force  $\mathbf{F}$  applied at  $B^*$ , but that  $\mathbf{F}$  is given by

$$\mathbf{F} = -\frac{GmM}{R^2} (\mathbf{a}_1 + \mathbf{f})$$

where

$$\begin{aligned} \mathbf{f} \triangleq \frac{3}{MR^2} \left\{ \frac{1}{2} [I_1(1 - 3C_{11}^2) + I_2(1 - 3C_{12}^2) + I_3(1 - 3C_{13}^2)]\mathbf{a}_1 \right. \\ + (I_1C_{21}C_{11} + I_2C_{22}C_{12} + I_3C_{23}C_{13})\mathbf{a}_2 \\ \left. + (I_1C_{31}C_{11} + I_2C_{32}C_{12} + I_3C_{33}C_{13})\mathbf{a}_3 \right\} \end{aligned}$$

with  $C_{ij} \triangleq \mathbf{a}_i \cdot \mathbf{b}_j$  ( $i, j = 1, 2, 3$ ). Find an expression for a contribution of  $\gamma$  to a potential energy of  $B$  in  $A$ .

**Result**

$$V_\gamma + \frac{GmM}{R} + \frac{Gm}{2R^3}(2I_1 - I_2 + I_3)$$

where  $V_\gamma$  is the potential energy contribution found in part (a) of Problem 9.9.

**9.11** A simple pendulum of mass  $m$  and length  $L$  is attached to a linear spring of natural length  $L'$  and spring constant  $k = 5mg/L$ , as shown in Fig. P9.11. A potential energy  $V$  of this system can be expressed as

$$V = mgL \sum_{i=1}^{\infty} a_i q^i$$

where  $a_1, a_2, \dots$  are constants. Determine  $a_1, \dots, a_4$ .

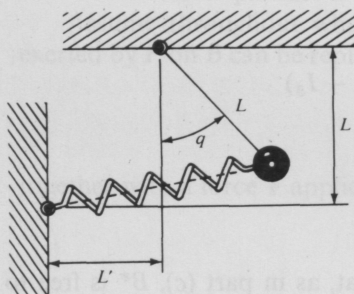


Figure P9.11

The generalized active force  $F$  corresponding to  $q$  can be expressed as

$$F = mgL \sum_{i=1}^{\infty} b_i q^i$$

where  $b_1, b_2, \dots$  are constants. Determine  $b_1, b_2$ , and  $b_3$ , both by using the potential energy  $V$  and without reference to potential energy, and comment briefly on the relative merits of the two methods.

**Results**  $a_1 = 0, a_2 = 3, a_3 = 0, a_4 = -\frac{7}{8}; b_1 = -6, b_2 = 0, b_3 = \frac{7}{2}$

**9.12** Three corners of a cube  $C$  are attached to fixed supports by means of identical, linear springs of spring constant  $k$ . When the springs are undeformed, their axes coincide with the edges of  $C$ , as indicated in Fig. P9.12.

To bring  $C$  into a general position, the center of  $C$  is displaced to a point whose coordinates relative to fixed axes  $X_1, X_2, X_3$  (see Fig. P9.12) are  $aq_1, aq_2, aq_3$ ,

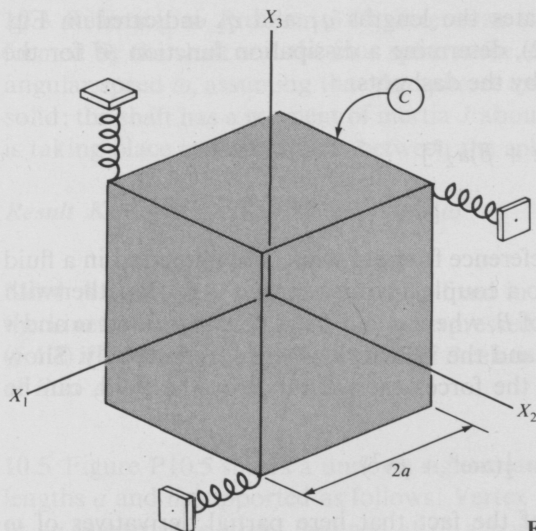


Figure P9.12

and three axes  $C_1, C_2, C_3$ , fixed in  $C$  and initially aligned with  $X_1, X_2, X_3$ , respectively, are brought into new orientations by means of successive right-handed rotations of  $C$  of amounts  $q_4$  about  $C_1$ ,  $q_5$  about  $C_2$ , and  $q_6$  about  $C_3$ .

Dropping all terms of third or higher degree in  $q_i$  ( $i = 1, \dots, 6$ ), find a potential energy contribution of the forces exerted on  $C$  by the springs.

**Result**  $\frac{1}{2}ka^2[(q_1 - q_5 - q_6)^2 + (q_2 - q_6 - q_4)^2 + (q_3 - q_4 - q_5)^2]$

**9.13** Two blocks are connected to each other and to a fixed support by means of springs and dashpots, as shown in Fig. P9.13. The springs have natural lengths  $L_1$  and  $L_2$ , and the force transmitted by each dashpot is proportional to the speed of the piston relative to the cylinder, the constants of proportionality having the values  $\alpha$  and  $\beta$ .

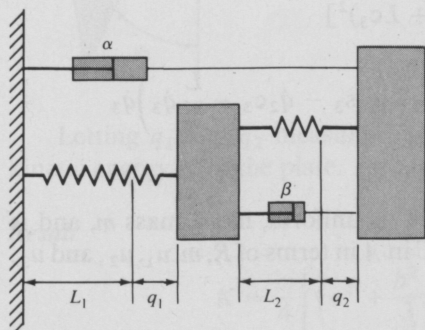


Figure P9.13

Using as generalized coordinates the lengths  $q_1$  and  $q_2$  indicated in Fig. P9.13, and letting  $u_r = \dot{q}_r$  ( $r = 1, 2$ ), determine a dissipation function  $\mathcal{F}$  for the set of forces exerted on the blocks by the dashpots.

**Result**  $\mathcal{F} = \frac{1}{2}[\alpha u_1^2 + 2\alpha u_1 u_2 + (\alpha + \beta)u_2^2]$

**\*9.14** A rigid body  $B$  moves in a reference frame  $A$  while  $B$  is immersed in a fluid that exerts on  $B$  forces equivalent to a couple having a torque  $-\alpha\omega$ , together with a force  $-\beta\mathbf{v}$ , applied at a point  $P$  of  $B$ , where  $\alpha$  and  $\beta$  are constants, and  $\omega$  and  $\mathbf{v}$  are the angular velocity of  $B$  in  $A$  and the velocity of  $P$  in  $A$ , respectively. Show that  $\mathcal{F}$ , a dissipation function for the forces exerted on  $B$  by the fluid, can be expressed as

$$\mathcal{F} = \frac{1}{2}(\alpha\omega^2 + \beta v^2)$$

*Suggestion:* Take advantage of the fact that here partial derivatives of  $\omega$  with respect to generalized speeds are partial angular velocities of  $B$  [see Eqs. (2.14.1)]; a similar statement applies to partial derivatives of  $\mathbf{v}$  and partial velocities of  $P$  [see Eqs. (2.14.2)].

## PROBLEM SET 10

(Secs. 5.4–5.6)

**10.1** Referring to the example in Sec. 2.9, and letting  $P_1$  and  $P_2$  have masses  $m_1$  and  $m_2$ , respectively, express the kinetic energies  ${}^A K^S$  and  ${}^B K^S$  of  $S$  in  $A$  and  $B$ , respectively, in terms of  $m_1$ ,  $m_2$ ,  $L$ ,  $\omega$ ,  $q_1$ ,  $q_2$ ,  $q_3$ , and the first time-derivatives of  $q_1$ ,  $q_2$ ,  $q_3$ , the generalized coordinates shown in Fig. 2.10.1.

**Results**

$${}^A K^S = {}^B K^S + \frac{1}{2}\omega^2[m_1 q_1^2 + m_2(q_1 + Lc_3)^2]$$

$${}^B K^S = \frac{1}{2}(m_1 + m_2)(\dot{q}_1^2 + \dot{q}_2^2) - m_2 L \left( \dot{q}_1 s_3 - \dot{q}_2 c_3 - \frac{L}{2} \dot{q}_3 \right) \dot{q}_3$$

**10.2** Referring to Problem 2.7, suppose that  $C$  is uniform, has a mass  $m$ , and is rolling on  $H$ . Express the kinetic energy  $K$  of  $C$  in  $A$  in terms of  $R$ ,  $m$ ,  $u_1$ ,  $u_2$ , and  $u_3$ .

**Result**  $K = (mR^2/8)(5u_1^2 + u_2^2 + 6u_3^2)$

**10.3** Referring to Problem 3.10, determine the kinetic energy  $K$  of the system formed by the shaft and the four spheres for an instant at which the shaft has an angular speed  $\omega$ , assuming that the spheres each have a mass  $m$ , are uniform and solid; the shaft has a moment of inertia  $J$  about its axis;  $\theta = 30^\circ$ ; and pure rolling is taking place at the contacts between the spheres and the race  $R$ .

**Result**  $K = \frac{1}{2}[J + 18mr^2(2 + \sqrt{3})/5]\omega^2$

**\*10.4** A point  $O$  of a rigid body  $B$  is fixed in a reference frame  $A$ . Show that  $K$ , the kinetic energy of  $B$  in  $A$ , can be expressed as  $K = K_\omega$ , with  $K_\omega$  given by Eq. (5.4.3) if  $\mathbf{I}$  denotes the inertia dyadic of  $B$  relative to  $O$  and  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $A$ .

**10.5** Figure P10.5 shows a uniform right-triangular plate of mass  $m$  and sides of lengths  $a$  and  $b$ , supported as follows: Vertex  $A$  is fixed and vertex  $B$  is attached to an inextensible string fastened at  $C$ , a point vertically above  $A$ , the length of the string being such that line  $AB$  is horizontal.

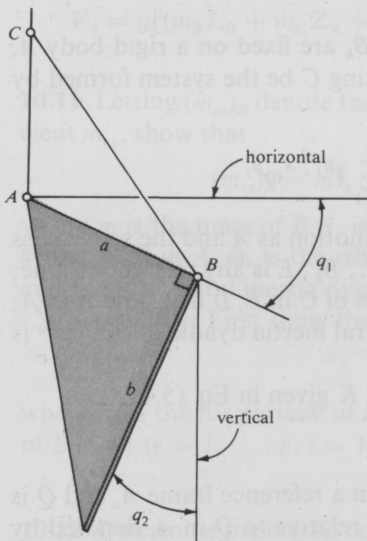


Figure P10.5

Letting  $q_1$  and  $q_2$  measure angles as indicated in Fig. P10.5, determine the kinetic energy  $K$  of the plate.

**Result**

$$K = \frac{m}{4} \left[ \left( a^2 + \frac{b^2}{3} s_2^2 \right) \dot{q}_1^2 + abc_2 \dot{q}_1 \dot{q}_2 + \frac{b^2}{3} \dot{q}_2^2 \right]$$

**\*10.6** At a certain instant, a rigid body  $B$  has a rotational kinetic energy  $K_\omega$  and a central angular momentum  $\mathbf{H}$ . Show that

$$\frac{1}{2} \frac{\mathbf{H}^2}{I_{\max}} \leq K_\omega \leq \frac{1}{2} \frac{\mathbf{H}^2}{I_{\min}}$$

where  $I_{\max}$  and  $I_{\min}$  are, respectively, the maximum and the minimum central moments of inertia of  $B$ .

**\*10.7** Letting  ${}^E K^C$ ,  ${}^E K^D$ , and  ${}^A K^B$  denote kinetic energies, show that

$${}^E K^C = {}^E K^D + {}^A K^B + \sum_{i=1}^{\beta} m_i {}^E \mathbf{v}^{\bar{A}_i} \cdot {}^A \mathbf{v}^{P_i}$$

if  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  ${}^E \mathbf{v}^{\bar{A}_i}$ , and  ${}^A \mathbf{v}^{P_i}$  are defined as follows:  $A$ , a rigid body;  $B$ , a set of  $\beta$  particles  $P_1, \dots, P_\beta$  of masses  $m_1, \dots, m_\beta$ , respectively;  $C$ , the system formed by  $A$  and  $B$ ;  $D$ , a rigid body that has the same motion as  $A$  and the same mass distribution as  $C$ ;  $E$ , a reference frame;  ${}^E \mathbf{v}^{\bar{A}_i}$ , the velocity in  $E$  of  $\bar{A}_i$ , the point of  $A$  with which  $P_i$  coincides;  ${}^A \mathbf{v}^{P_i}$ , the velocity of  $P_i$  in  $A$ .

**\*10.8** The mass centers of  $k$  rigid bodies  $B_1, \dots, B_k$  are fixed on a rigid body  $A$ , but  $B_k$  is otherwise free to move relative to  $A$ . Letting  $C$  be the system formed by  $A$  and  $B_1, \dots, B_k$ , show that

$${}^E K^C = {}^E K^D + {}^A K^B + {}^E \boldsymbol{\omega}^A \cdot \sum_{j=1}^k \mathbf{I}^{B_j} \cdot {}^A \boldsymbol{\omega}^{B_j}$$

where  $D$  designates a rigid body that has the same motion as  $A$  and the same mass distribution as  $C$ ;  $B$  is the set of rigid bodies  $B_1, \dots, B_k$ ;  $E$  is any reference frame;  ${}^E K^C$ ,  ${}^E K^D$ ,  ${}^A K^B$  are, respectively, the kinetic energies of  $C$  in  $E$ ,  $D$  in  $E$ , and  $B$  in  $A$ ;  ${}^E \boldsymbol{\omega}^A$  is the angular velocity of  $A$  in  $E$ ;  $\mathbf{I}^{B_j}$  is the central inertia dyadic of  $B_j$ ;  ${}^A \boldsymbol{\omega}^{B_j}$  is the angular velocity of  $B_j$  in  $A$ .

Use this result to determine the kinetic energy  $K$  given in Eq. (5.4.28).

**10.9** If  $S$  is a set of  $v$  particles  $P_1, \dots, P_v$  moving in a reference frame  $A$ , and  $Q$  is a point moving in  $A$ , then the kinetic energy of  $S$  relative to  $Q$  in  $A$ , denoted by  $K^{S/Q}$ , is defined as

$$K^{S/Q} \triangleq \frac{1}{2} \sum_{i=1}^v m_i \mathbf{v}^{P_i/Q}$$

where  $m_i$  is the mass of  $P_i$ , and  $\mathbf{v}^{P_i/Q}$ , called the velocity of  $P_i$  relative to  $Q$  in  $A$ , is the difference between  $\mathbf{v}^{P_i}$ , the velocity of  $P_i$  in  $A$ , and  $\mathbf{v}^Q$ , the velocity of  $Q$  in  $A$ ; that is

$$\mathbf{v}^{P_i/Q} \triangleq \mathbf{v}^{P_i} - \mathbf{v}^Q \quad (i = 1, \dots, v)$$

Letting  $S^*$  be the mass center of  $S$ , show that  $K^S$ , the kinetic energy of  $S$  in  $A$ , is given by

$$K^S = K^{S/S^*} + K^{S^*}$$

where  $K^{S^*}$  is the kinetic energy in  $A$  of a particle whose mass is equal to the total mass of  $S$  and whose velocity in  $A$  is equal to the velocity of  $S^*$  in  $A$ .

Comment on the relationship between these facts and Eqs. (5.4.2)–(5.4.4).

**\*10.10** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, and 8.21), determine  $K$ , the kinetic energy of the robot arm in  $E$ , and  $V_\gamma$ , a potential energy contribution of the set of gravitational forces acting on the robot arm.

### Results

$$\begin{aligned} K = & \frac{1}{2}[A_1 u_1^2 + (B_1 + C_1)Z_1^2 + (B_2 + C_2)u_2^2 + (B_3 + C_3)Z_2^2 \\ & + Z_1(D_{11}Z_1 + D_{12}u_2 + D_{13}Z_2) + u_2(D_{21}Z_1 + D_{22}u_2 + D_{23}Z_2) \\ & + Z_2(D_{31}Z_1 + D_{32}u_2 + D_{33}Z_2) + m_A Z_5^2 + m_B(Z_7^2 + Z_8^2) \\ & + m_C(Z_{11}^2 + Z_{12}^2 + u_3^2) + m_D(Z_{17}^2 + Z_{18}^2 + Z_{19}^2)] \\ V_\gamma = & g[(m_B L_B + m_C Z_9 + m_D Z_{14})s_1 + m_D p_1 c_1] \end{aligned}$$

**10.11** Letting  $(m_{rs})_B$  denote the contribution of a rigid body  $B$  to the inertia coefficient  $m_{rs}$ , show that

$$(m_{rs})_B = m \tilde{\mathbf{v}}_r \cdot \tilde{\mathbf{v}}_s + \tilde{\boldsymbol{\omega}}_r \cdot \mathbf{I} \cdot \tilde{\boldsymbol{\omega}}_s \quad (r, s = 1, \dots, p)$$

where  $m$  is the mass of  $B$ ,  $\tilde{\mathbf{v}}_r$  is the  $r$ th nonholonomic partial velocity of the mass center of  $B$  in  $A$ ,  $\tilde{\boldsymbol{\omega}}_r$  is the  $r$ th nonholonomic partial angular velocity of  $B$  in  $A$ , and  $\mathbf{I}$  is the central inertia dyadic of  $A$ .

*Suggestion:* First show that

$$\tilde{\mathbf{v}}_r^{P_i} = \tilde{\mathbf{v}}_r + \tilde{\boldsymbol{\omega}}_r \times \mathbf{r}_i$$

where  $P_i$  is the  $i$ th particle of  $B$  and  $\mathbf{r}_i$  is the position vector from the mass center of  $B$  to  $P_i$  ( $r = 1, \dots, p; i = 1, \dots, v$ ).

**10.12** Determine the inertia coefficients for the system considered in the examples in Secs. 4.8 and 5.4.

### Results

$$m_{11} = I_A + m_A a^2 + m_B \left( \frac{R^2}{2} + 3b^2 \right)$$

$$m_{22} = m_A + 3m_B$$

$$m_{12} = m_{21} = 0$$

**10.13** When the inertia coefficient  $m_{rs}$  differs from zero, *dynamic coupling* is said to exist between the generalized speeds  $u_r$  and  $u_s$ .

Suppose that the mass center of a rigid body  $B$  is fixed in a reference frame  $A$ , and let  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  be mutually perpendicular unit vectors parallel to central principal axes of  $B$ . Bring  $B$  into a general orientation in  $A$  by aligning  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  with  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , respectively, where  $\mathbf{a}_i$  ( $i = 1, 2, 3$ ) is a unit vector fixed in  $A$ , and then subjecting  $B$  to successive rotations characterized by the vectors  $q_1\mathbf{b}_1, q_2\mathbf{b}_2, q_3\mathbf{b}_3$ . Define generalized speeds  $u_1, u_2, u_3$  as (a)  $u_r \triangleq \dot{q}_r$  ( $r = 1, 2, 3$ ) and (b)  $u_r \triangleq \boldsymbol{\omega} \cdot \mathbf{b}_r$  ( $r = 1, 2, 3$ ), where  $\boldsymbol{\omega}$  is the angular velocity of  $B$  in  $A$ . Determine which of the generalized speeds are coupled dynamically.

**Results** (a)  $u_1$  and  $u_2, u_1$  and  $u_3$ ; (b) None

**10.14** Referring to Problem 8.16, determine  $\tilde{F}_2^*$  by using Eqs. (5.6.6). Comment briefly on the relative merits of using Eqs. (5.6.6), on the one hand, and Eqs. (4.11.1), on the other hand, to determine  $\tilde{F}_2^*$ .

**10.15** For the system considered in Problem 8.7, determine the generalized inertia force  $F_1^*$  by using (a) Eqs. (4.11.7), (b) Eqs. (5.6.8), and (c) the formula given in Problem 8.22. Comment briefly on the relative merits of the three methods.

**Result**  $m[-29\dot{u}_1 + 19(c_1c_2 - s_1s_2)\dot{u}_2 - 8(s_3s_1 + c_3c_1)\dot{u}_3$   
 $- 19(s_2c_1 + c_2s_1)(u_2^2/L) - 8(s_1c_3 - c_1s_3)(u_3^2/L)]$

**\*10.16** A rigid body  $B$  forms a portion of a simple nonholonomic system consisting of  $\nu$  particles. Letting  $m$  be the mass,  $\mathbf{I}$  the central inertia dyadic,  $\mathbf{v}$  the velocity of the mass center, and  $\boldsymbol{\omega}$  the angular velocity of  $B$ , show that the contribution of  $B$  to the left-hand member of Eq. (5.6.1) is  $m\mathbf{v} \cdot \dot{\tilde{\mathbf{v}}}_t + \boldsymbol{\omega} \cdot \mathbf{I} \cdot \dot{\tilde{\boldsymbol{\omega}}}_t$ , where  $\tilde{\mathbf{v}}_t$  and  $\tilde{\boldsymbol{\omega}}_t$  have the same meanings as in Eqs. (2.14.3) and (2.14.4), and the dots over these symbols denote time-differentiation in  $A$ .

## PROBLEM SET 11

(Secs. 6.1–6.3)

**11.1** Referring to the example in Sec. 2.9 (see also Problems 8.1 and 8.15), let line  $Y$  be vertical, let  $P_1$  and  $P_2$  have masses  $m_1$  and  $m_2$ , respectively, and assume that the mass of  $R$  is negligible. Letting  $A$  be fixed relative to the Earth, and treating the Earth as a Newtonian reference frame, determine  $f_1, f_2$ , and  $f_3$  such that the dynamical equations governing  $u_1, u_2$ , and  $u_3$ , defined as

$$u_1 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_x \quad u_2 \triangleq {}^A\mathbf{v}^{P_1} \cdot \mathbf{e}_y \quad u_3 \triangleq \dot{q}_3$$



where  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are unit vectors directed as shown in Fig. 2.6.1, can be expressed as

$$\dot{u}_i = f_i \quad (i = 1, 2, 3)$$

### Results

$$\begin{aligned} f_1 &= -gs_3 + \omega^2 q_1 c_3 + u_2 u_3 + (\omega^2 c_3^2 + u_3^2) L m_2 / (m_1 + m_2) \\ f_2 &= -gc_3 - (\omega^2 q_1 s_3 + u_3 u_1) \\ f_3 &= -\omega^2 s_3 c_3 \end{aligned}$$

**11.2** For the system  $S$  formed by the particle  $P_1$ , disk  $D$ , and rod  $R$  considered in Problems 8.2 and 8.16, make the same assumptions as in Problem 11.1 regarding the line  $Y$  and the reference frame  $A$ . Determine  $f_1$  and  $f_2$  such that the dynamical equations governing  $u_1$  and  $u_2$  can be expressed as

$$\dot{u}_i = f_i \quad (i = 1, 2)$$

### Results

$$\begin{aligned} f_1 &= -gs_3 + \omega^2 q_1 c_3 + \frac{1}{m_1 + m_2} \left( m_2 L \omega^2 c_3^2 - \frac{m_1}{L} u_2^2 \right) \\ f_2 &= -gc_3 - \omega^2 q_1 s_3 + \frac{u_1 u_2}{L} \end{aligned}$$

**11.3** Regarding the Earth  $E$ , Moon  $M$ , and Sun  $S$  each as a particle, let  $Q$  be the mass center of  $E$  and  $S$ , and let  $F_E$ ,  $F_M$ ,  $F_S$ , and  $F_Q$  be reference frames in which  $E$ ,  $M$ ,  $S$ , and  $Q$ , respectively, are fixed and whose relative orientations do not vary with time. Furthermore, assume that  $F_S$  can be chosen in such a way that  $E$  moves in  $F_S$  on a circular path centered at  $S$  and traced out once per year.

Assuming that  $F_Q$  is a Newtonian reference frame, assess the advisability of regarding  $F_E$ ,  $F_M$ , and  $F_S$  as Newtonian reference frames for the purpose of analyzing motions of  $E$ ,  $M$ ,  $e$ , and  $m$ , where  $e$  and  $m$  designate a low-altitude satellite of  $E$  and a low-altitude satellite of  $M$ , respectively. In making these assessments, assume that  $S$ ,  $E$ ,  $M$ ,  $e$ , and  $m$  are at all times coplanar, and let the distances and angular speeds indicated in Fig. P11.3 have the following values:

$$\begin{aligned} R_{SQ} &= 4.5 \times 10^5 \text{ m} & R_{QE} &= 1.5 \times 10^{11} \text{ m} \\ R_{EM} &= 4.0 \times 10^8 \text{ m} & R_{Ee} &= 7.0 \times 10^6 \text{ m} \\ R_{Mm} &= 2.0 \times 10^6 \text{ m} \\ \omega_E &= 2 \times 10^{-7} \text{ rad/s} & \omega_M &= 24 \times 10^{-7} \text{ rad/s} \\ \omega_e &= 12 \times 10^{-4} \text{ rad/s} & \omega_m &= 10 \times 10^{-4} \text{ rad/s} \end{aligned}$$

**Results** Table P11.3

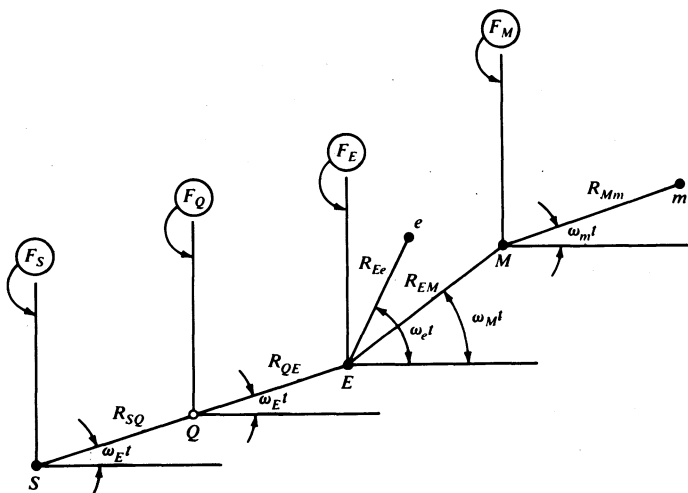


Figure P11.3

Table P11.3

Moving object	Approximately Newtonian reference frame
$E$	$F_S$
$M$	$F_S$
$e$	$F_S, F_E, F_M$
$m$	$F_S, F_E, F_M$

11.4 The dynamical equations governing the generalized speeds  $u_1$  and  $u_2$  introduced in the example in Sec. 4.8 are to be formulated under the assumptions made in Problems 8.12 and 8.17. Determine  $f_1$  and  $f_2$  such that these equations can be expressed as

$$\dot{u}_i = f_i \quad (i = 1, 2)$$

### Results

$$f_1 = \frac{f Q_3 + M g e \sin \theta \cos q_1 - m_A a u_1 u_2}{I_A + 2 J b^2 / R^2 + 2 K + m_A a^2 + 2 m_B b^2}$$

$$f_2 = \frac{Q_2 + M g \sin \theta \sin q_1 + m_A a u_1^2}{m_A + 2 m_B + 2 J / R^2}$$

where

$$Q_2 = \frac{-\mu' u_2 Q_1}{(u_2^2 + f^2 u_1^2)^{1/2}}$$

$$Q_3 = \frac{-\mu' u_1 f Q_1}{(u_2^2 + f^2 u_1^2)^{1/2}}$$

with

$$Q_1 = \frac{eMg \cos \theta}{f - \mu' R u_2 / (u_2^2 + f^2 u_1^2)^{1/2}}$$

**11.5** Referring to Problem 2.7 and letting  $C$  be a uniform disk *rolling* on  $H$ , determine  $f_1$ ,  $f_2$ , and  $f_3$  such that the dynamical equations governing  $u_1$ ,  $u_2$ , and  $u_3$  as defined in Problem 8.3 can be written

$$\dot{u}_i = f_i \quad (i = 1, 2, 3)$$

**Results**

$$f_1 = \frac{1}{5}(u_2^2 \tan q_2 - 6u_2 u_3 - 4gs_2/R)$$

$$f_2 = 2u_3 u_1 - u_1 u_2 \tan q_2$$

$$f_3 = \frac{2}{3}u_1 u_2$$

**11.6** A couple of torque  $T\mathbf{a}_z$  is applied to the link of length  $L_1$  of Problem 4.3. Determine  $f_1$  such that the dynamical equation governing  $u_1$  as defined in Problem 8.6 can be written  $\dot{u}_1 = f_1$ .

**Result**

$$\begin{aligned} f_1 = & \left\{ [T + (\tilde{F}_1)_y] \sin^2(q_2 - q_3) \right. \\ & + m_2 L_1 L_3 \sin(q_1 - q_2) \left[ \frac{L_1}{L_3} u_1 (u_2 - u_1) \cos(q_1 - q_2) \right. \\ & \left. \left. + u_3 (u_3 - u_2) \cos(q_2 - q_3) \right] \right\} / \left\{ \left[ m_1 \sin^2(q_3 - q_2) \right. \right. \\ & \left. \left. + m_2 \sin^2(q_1 - q_2) \right] L_1^2 \right\} \end{aligned}$$

where  $(\tilde{F}_1)_y$  is the generalized active force contribution found in Problem 8.6.

**\*11.7** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, 8.21 and 10.10), show that the dynamical equations of motion of the robot arm can be expressed as

$$\sum_{r=1}^3 X_{sr} \dot{u}_r = Y_s \quad (s = 1, 2, 3)$$

where  $X_{rs} = X_{sr}$  ( $r, s = 1, 2, 3; r \neq s$ ). Determine  $X_{rs}$  and  $Y_s$  ( $r, s = 1, 2, 3$ ).

### Results

$$X_{11} = -[A_1 + c_1(Z_{35} + Z_{41} + Z_{49}) + s_1(Z_{39} + Z_{44} + Z_{53}) + m_A L_A^2 + m_B Z_6^2 + m_C Z_{10}^2 + m_D(Z_{13}^2 + Z_{15}^2 + Z_{16}^2)]$$

$$X_{12} = X_{21} = -[Z_{51} + m_D(Z_{13}Z_{14} - Z_{16}p_1)]$$

$$X_{13} = X_{31} = -m_D Z_{16}$$

$$X_{22} = -[B_2 + C_2 + D_{22} + m_B L_B^2 + m_C Z_9^2 + m_D(Z_{14}^2 + p_1^2)]$$

$$X_{23} = X_{32} = m_D p_1$$

$$X_{33} = -(m_C + m_D)$$

$$Y_1 = c_1(Z_{36} + Z_{42} + Z_{50}) + s_1(Z_{40} + Z_{45} + Z_{54}) + m_B Z_6 Z_{23} + m_C Z_{10} Z_{27} + m_D(Z_{13}Z_{32} + Z_{15}Z_{33} + Z_{16}Z_{34}) - T_1^{E/A}$$

$$Y_2 = Z_{38} + Z_{43} + Z_{52} + m_B L_B Z_{22} + m_C Z_9 Z_{26} + m_D(Z_{14}Z_{32} - p_1 Z_{34}) - T_2^{A/B} + g[(m_B L_B + m_C Z_9 + m_D Z_{14})c_1 - m_D p_1 s_1]$$

$$Y_3 = m_C Z_{28} + m_D Z_{34} - K_3^{B/C} + g(m_C + m_D)s_1$$

**\*11.8** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, 8.21, 10.10, and 11.7), suppose that  $D^*$  lies on line  $B^*C^*$  and that each central principal axis of  $D$  is parallel to one of  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ . Show that under these circumstances the dynamical equations of motion reduce to

$$\dot{u}_r = \frac{Y_r}{X_{rr}} \quad (r = 1, 2, 3)$$

**\*11.9** Show that Eqs. (6.1.1) can be written

$$F_r + F_r^* + \sum_{s=p+1}^n (F_s + F_s^*) A_{sr} = 0 \quad (r = 1, \dots, p)$$

where  $F_1, \dots, F_n$  are *holonomic* generalized active forces for  $S$  in  $N$ ,  $F_1^*, \dots, F_n^*$  are *holonomic* generalized inertia forces for  $S$  in  $N$ , and  $A_{p+1,1}, \dots, A_{n,p}$  have the same meaning as in Eqs. (2.13.1).

**11.10** Figure P11.10 represents a one-cylinder reciprocating engine consisting of a counter-weighted crank  $A$ , connecting rod  $B$ , piston  $C$ , and cylinder  $D$ .  $A^*$ ,  $B^*$ , and  $C^*$  are the mass centers of  $A$ ,  $B$ , and  $C$ , respectively, and  $A$ ,  $B$ , and  $C$  have masses  $m_A$ ,  $m_B$ , and  $m_C$ , respectively. The central radii of gyration of  $A$  and  $B$  with respect to axes normal to the middle plane of the mechanism have the values  $k_A$  and  $k_B$ , respectively.

The system  $S$  formed by  $A$ ,  $B$ , and  $C$  possesses one degree of freedom in a reference frame in which point  $O$  and the axis of the cylinder are fixed; the angle  $q_1$  shown in Fig. P11.10, and  $u_1$  defined as  $u_1 \triangleq \dot{q}_1$ , can serve, respectively, as a generalized coordinate and as a generalized speed associated with this degree of freedom. However, it can be convenient, for example, for the purpose of bringing certain interaction forces into evidence (see Sec. 4.9), to introduce additional generalized speeds,  $u_2$  and  $u_3$ , as

$$u_2 \triangleq \boldsymbol{\omega}^B \cdot \mathbf{n}_2 \quad u_3 \triangleq \mathbf{v}^{C^*} \cdot \mathbf{n}_3$$

Moreover, it is then helpful to define, in addition, an angle  $q_2$  as indicated in Fig. P11.10.

Let the set of contact forces exerted on  $A$  by the crankshaft on which  $A$  is mounted be equivalent to a couple of torque  $\alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2 + \alpha_3 \mathbf{n}_3$ , together with a force  $P_1 \mathbf{n}_1 + P_2 \mathbf{n}_2 + P_3 \mathbf{n}_3$  applied at  $O$ , and let the set of contact forces exerted on  $C$  by exploding gases and by the cylinder  $D$  be equivalent to a couple of torque  $\beta_1 \mathbf{n}_1 + \beta_2 \mathbf{n}_2 + \beta_3 \mathbf{n}_3$ , together with a force  $Q_1 \mathbf{n}_1 + Q_2 \mathbf{n}_2 + Q_3 \mathbf{n}_3$  applied at  $C^*$ . Finally, let the set of forces exerted by  $B$  on  $A$  be equivalent to a couple of torque  $\gamma_1 \mathbf{n}_1 + \gamma_3 \mathbf{n}_3$ , together with a force  $R_1 \mathbf{n}_1 + R_2 \mathbf{n}_2 + R_3 \mathbf{n}_3$  applied at point  $P$ .

After verifying that  $u_2$  and  $u_3$  must satisfy the motion constraint equations

$$u_2 = -\frac{ac_1}{bc_2} u_1$$

$$u_3 = -\frac{a}{c_2} (s_1 c_2 + c_1 s_2) u_1$$

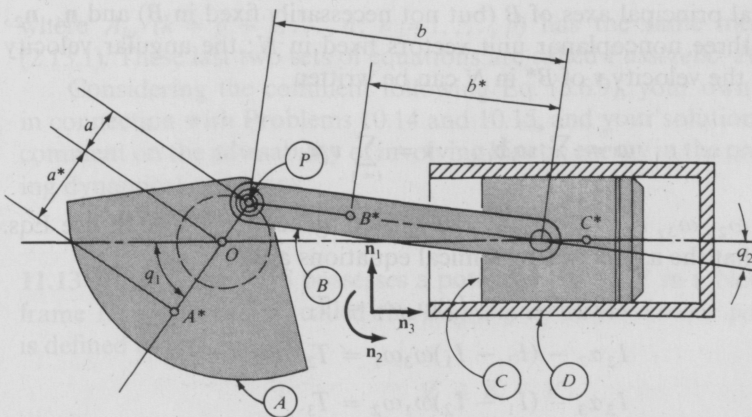


Figure P11.10

write dynamical equations governing all motions of  $S$  (a) by using Eqs. (6.1.2) with  $r = 1, 2, 3$  and (b) by using the equation in Problem 11.9 with  $p = 1$ . Neglect gravitational forces, and verify that the result of part (b) can be obtained from those of part (a) by using the first two equations of part (a) to eliminate  $R_3$  from the third equation.

### Results

$$\begin{aligned}
 (a) \quad & m_A[(a^*)^2 + k_A^2]\dot{u}_1 = \alpha_2 + a(R_1c_1 - R_3s_1) \\
 & m_B\{[(b^*)^2 + k_B^2]\dot{u}_2 - b^*s_2\dot{u}_3\} = b(R_1c_2 + R_3s_2) \\
 & (m_B + m_C)\dot{u}_3 - m_Bb^*s_2\dot{u}_2 = Q_3 - R_3 - m_Bb^*c_2u_2^2 \\
 (b) \quad & m_A[(a^*)^2 + k_A^2]\dot{u}_1 + \frac{m_Ba}{c_2} \left\langle \left\{ b^*s_2(s_1c_2 + c_1s_2) - c_1 \left[ \frac{(b^*)^2 + k_B^2}{b} \right] \right\} \dot{u}_2 \right. \\
 & \quad \left. + \left[ -\frac{m_C}{m_B}(s_1c_2 + c_1s_2) + \left( \frac{b^*}{b} - 1 \right) c_1s_2 - s_1c_2 \right] \dot{u}_3 \right\rangle \\
 & = \alpha_2 + a(s_1c_2 + c_1s_2) \left( m_Bb^*u_2^2 - \frac{Q_3}{c_2} \right)
 \end{aligned}$$

**11.11** When a rigid body  $B$  moves in a Newtonian reference frame  $N$  under the action of a set of contact and distance forces equivalent to a couple of torque  $\mathbf{T}$  together with a force  $\mathbf{F}$  applied at the mass center  $B^*$  of  $B$ , the vectors  $\mathbf{T}$  and  $\mathbf{F}$  always can be expressed as

$$\mathbf{T} = \sum_{i=1}^3 T_i \mathbf{b}_i \quad \mathbf{F} = \sum_{i=1}^3 F_i \mathbf{n}_i$$

where  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_3$  form a dextral set of mutually perpendicular unit vectors parallel to central principal axes of  $B$  (but not necessarily fixed in  $B$ ) and  $\mathbf{n}_1, \mathbf{n}_2$ , and  $\mathbf{n}_3$  are any three noncoplanar unit vectors fixed in  $N$ ; the angular velocity  $\boldsymbol{\omega}$  of  $B$  in  $N$  and the velocity  $\mathbf{v}$  of  $B^*$  in  $N$  can be written

$$\boldsymbol{\omega} = \sum_{i=1}^3 \omega_i \mathbf{b}_i \quad \mathbf{v} = \sum_{i=1}^3 v_i \mathbf{n}_i$$

Letting  $\omega_1, \omega_2, \omega_3, v_1, v_2, v_3$  play the roles of generalized speeds, use Eqs. (6.1.2) to show that the associated dynamical equations are

$$I_1\alpha_1 - (I_2 - I_3)\omega_2\omega_3 = T_1$$

$$I_2\alpha_2 - (I_3 - I_1)\omega_3\omega_1 = T_2$$

$$I_3\alpha_3 - (I_1 - I_2)\omega_1\omega_2 = T_3$$

$$F_1 = m\dot{v}_1 \quad F_2 = m\dot{v}_2 \quad F_3 = m\dot{v}_3$$

where  $I_j$  is the moment of inertia of  $B$  about a line passing through  $B^*$  and parallel to  $\mathbf{b}_i$ , and  $\alpha_i \triangleq \boldsymbol{\alpha} \cdot \mathbf{b}_i$ , with  $\boldsymbol{\alpha}$  the angular acceleration of  $B$  in  $N$ . The first three equations are known as *Euler's dynamical equations*, and the last three equations express *Newton's second law* of motion for the mass center of a rigid body.

**11.12** Referring to Sec. 5.6, show that one can use Eqs. (6.1.1) or Eqs. (6.1.2) to generate dynamical equations involving the kinetic energy  $K$  of a system  $S$  in a Newtonian reference frame, as follows.

For a holonomic system with  $u_r \triangleq \dot{q}_r$  ( $r = 1, \dots, n$ ),

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} = F_r \quad (r = 1, \dots, n)$$

These equations are known as *Lagrange's equations of the first kind*.

For a holonomic system with  $u_r$  defined as in Eqs. (2.12.1), so that Eqs. (2.14.5) apply,

$$\sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) W_{sr} = F_r \quad (r = 1, \dots, n)$$

For a nonholonomic system with  $u_r \triangleq \dot{q}_r$  ( $r = 1, \dots, n$ ),

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} + \sum_{s=p+1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) C_{sr} = \tilde{F}_r \quad (r = 1, \dots, p)$$

where  $C_{sr}$  ( $s = p+1, \dots, n$ ;  $r = 1, \dots, p$ ) has the same meaning as in Eqs. (5.1.13).

For a nonholonomic system with  $u_r$  defined as in Eqs. (2.12.1), so that Eqs. (2.14.5) apply,

$$\sum_{s=1}^n \left( \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_s} - \frac{\partial K}{\partial q_s} \right) \left( W_{sr} + \sum_{k=p+1}^n W_{sk} A_{kr} \right) = \tilde{F}_r \quad (r = 1, \dots, p)$$

where  $A_{kr}$  ( $k = p+1, \dots, n$ ;  $r = 1, \dots, p$ ) has the same meaning as in Eqs. (2.13.1). These last two sets of equations are called *Passerello-Huston equations*.

Considering the comment following Eq. (5.6.9), your own comments made in connection with Problems 10.14 and 10.15, and your solution of Problem 4.18, comment on the advisability of involving kinetic energy in the process of formulating dynamical equations.

**11.13** When a system  $S$  possesses a potential energy  $V$  in a Newtonian reference frame  $N$ , a quantity  $\mathcal{L}$ , called the *Lagrangian* or the *kinetic potential* of  $S$  in  $N$ , is defined as

$$\mathcal{L} \triangleq K - V$$

where  $K$  is the kinetic energy of  $S$  in  $N$ .

Supposing that  $S$  is a holonomic system with generalized speeds defined as  $u_r \triangleq \dot{q}_r$  ( $r = 1, \dots, n$ ), where  $q_1, \dots, q_n$  are generalized coordinates for  $S$  in  $N$ , show that  $\mathcal{L}$ , regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and the time  $t$ , satisfies the equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}}{\partial q_r} = 0 \quad (r = 1, \dots, n)$$

These equations are called *Lagrange's equations of the second kind*.

**11.14** Referring to the example in Sec. 4.8 (see also Problems 8.12 and 8.17), express  $Q_1$  as a function of  $\mu', R, M, g, e, f, \theta, u_1$ , and  $u_2$ .

**Result**  $Q_1 = eMg \cos \theta / [f - \mu' R u_2 / (u_2^2 + f^2 u_1^2)^{1/2}]$

## PROBLEM SET 12

(Secs. 6.4–6.7)

**12.1** Referring to the discussion in Sec. 6.4 of motions of the bar  $B$  depicted in Fig. 6.4.3, suppose that  $h = 3R/4$  and  $\Omega^2 = 2g/R$ . Find a value of  $\bar{q}_1$  other than zero such that the equation  $q_1 = \bar{q}_1$  describes a possible motion of  $B$ , and determine  $N$  such that the circular frequency of small amplitude oscillations of  $B$  that ensue subsequent to a small disturbance of this motion is equal to  $N(g/R)^{1/2}$ .

**Results**  $\bar{q}_1 = \cos^{-1}(9/10) \quad N = (19/85)^{1/2}$

**12.2** In Fig. P12.2,  $A$ ,  $B$ , and  $C$  are the outer gimbal, the inner gimbal, and the rotor of a gyroscope, and  $P$  is a particle of mass  $m$  attached to the rotor axis at a distance  $h$  from the center of  $C$ . By means of an electric motor (not shown) that may be regarded as consisting of two parts, one rigidly attached to  $B$ , the other to  $C$ ,  $C$  is driven relative to  $B$  in such a way that the angular velocity of  $C$  in  $B$  is given by

$${}^B\boldsymbol{\omega}^C = s\mathbf{b}_1$$

where  $s$  is a constant and  $\mathbf{b}_1$  is a unit vector directed as shown. The inertia properties of  $A$ ,  $B$ , and  $C$  are characterized by the five quantities  $I, J_1, J_2, K_1$ , and  $K_2$  defined in terms of the central inertia dyadics  $\mathbf{I}_A, \mathbf{I}_B$ , and  $\mathbf{I}_C$  of  $A$ ,  $B$ , and  $C$ , respectively, as follows:

$$I \triangleq \mathbf{a}_1 \cdot \mathbf{I}_A \cdot \mathbf{a}_1 \quad J_r \triangleq \mathbf{b}_r \cdot \mathbf{I}_B \cdot \mathbf{b}_r \quad K_r \triangleq \mathbf{b}_r \cdot \mathbf{I}_C \cdot \mathbf{b}_r \quad (r = 1, 2)$$





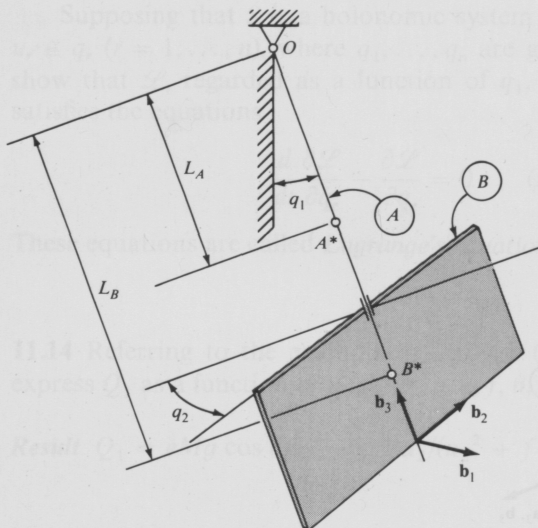


Figure P12.3

are  $L_A$  and  $L_B$ , respectively. Finally,  $A$  has a moment of inertia  $A_1$  about any line normal to  $A$  and passing through  $A^*$ , and the central principal moments of inertia of  $B$  have the values  $B_1$ ,  $B_2$ , and  $B_3$ , where the subscripts refer to the unit vectors  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , and  $\mathbf{b}_3$  shown in Fig. P12.3.

$S$  can move in such a way that the angles  $q_1$  and  $q_2$  (see Fig. P12.3) are given for all values of the time  $t$  by

$$q_1 = \tilde{q}_1 \quad q_2 = 0$$

where  $\tilde{q}_1$  is a function of  $t$ . Letting  $q_1^*$  and  $q_2^*$  be perturbations such that

$$q_1 = \tilde{q}_1 + q_1^* \quad q_2 = q_2^*$$

formulate differential equations linearized in  $q_1^*$ ,  $q_2^*$ ,  $\dot{q}_1^*$ , and  $\dot{q}_2^*$ . Use these equations to verify that  $\tilde{q}_1$ ,  $q_1^*$ , and  $q_2^*$  satisfy the differential equations

$$\ddot{\tilde{q}}_1 + \alpha \sin \tilde{q}_1 \approx 0$$

$$\ddot{q}_1^* + \alpha \cos \tilde{q}_1 q_1^* \approx 0 \quad \ddot{q}_2^* - \beta \dot{\tilde{q}}_1^2 q_2^* \approx 0$$

where  $\alpha$  and  $\beta$  are constants, and determine these constants. Answer the following questions regarding the differential equations: (a) Are all of the equations linear differential equations? (b) Are the last two equations linear differential equations? (c) Is one or more of the equations a differential equation with constant coefficients? (d) Are the last two equations differential equations with constant coefficients?

### Results

$$\alpha = \frac{g(m_A L_A + m_B L_B)}{A_1 + B_1 + m_A L_A^2 + m_B L_B^2} \quad \beta = \frac{B_2 - B_1}{B_3}$$

(a) No; (b) Yes; (c) Yes; (d) No

**12.4** Consider a holonomic system  $S$  that consists of particles  $P_1, \dots, P_v$  and possesses generalized coordinates  $q_1, \dots, q_n$  in a Newtonian reference frame  $N$ . Define generalized speeds  $u_1, \dots, u_n$  for  $S$  in  $N$  as

$$u_r = \dot{q}_r \quad (r = 1, \dots, n)$$

and suppose that when  $\mathbf{v}^{P_i}$ , the velocity of  $P_i$  in  $N$ , is expressed as [see Eq. (2.14.2)]

$$\mathbf{v}^{P_i} = \sum_{r=1}^n \mathbf{v}_r^{P_i} u_r + \mathbf{v}_i^{P_i} \quad (i = 1, \dots, v)$$

then  $\mathbf{v}_i^{P_i} = 0$  ( $i = 1, \dots, v$ ) and  $\mathbf{v}_r^{P_i}$  ( $i = 1, \dots, v; r = 1, \dots, n$ ) depends on the time  $t$  solely because  $q_1, \dots, q_n$  depend on  $t$ . Finally, assume that  $S$  possesses a potential energy  $V$  in  $N$ , that  $S$  can remain at rest in  $N$ , and that  $q_1, \dots, q_n$  have been chosen in such a way that, when  $S$  is at rest in  $N$ , the dynamical equations governing all motions of  $S$  in  $N$  are satisfied by  $q_r \equiv 0$  ( $r = 1, \dots, n$ ). Show that under these circumstances the dynamical equations of  $S$  in  $N$ , when linearized in  $q_r$  and  $\dot{q}_r$  ( $r = 1, \dots, n$ ), can be written

$$\sum_{s=1}^n (M_{rs} \ddot{q}_s + K_{rs} q_s) \approx 0 \quad (r = 1, \dots, n)$$

where  $M_{rs}$  ( $r, s = 1, \dots, n$ ) are the values of the inertia coefficients of  $S$  in  $N$  (see Sec. 5.5) when  $q_1 = \dots = q_n = 0$ , and  $K_{rs}$  denotes the value of  $\partial^2 V / \partial q_r \partial q_s$  ( $r, s = 1, \dots, n$ ) when  $q_1 = \dots = q_n = 0$ .

*Suggestion:* Refer to Eq. (5.1.2) for  $F_r$  ( $r = 1, \dots, n$ ), and make use of the facts that

$$\frac{\partial V}{\partial q_r} = \frac{\partial V(0)}{\partial q_r} + \sum_{s=1}^n \frac{\partial^2 V(0)}{\partial q_r \partial q_s} q_s + \dots \quad (r = 1, \dots, n)$$

while

$$\mathbf{v}_r^{P_i} = \mathbf{v}_r^{P_i}(0) + \sum_{s=1}^n \frac{\partial \mathbf{v}_r^{P_i}(0)}{\partial q_s} q_s + \dots \quad (i = 1, \dots, v; r = 1, \dots, n)$$

from which it follows that the linearized velocity of  $P_i$  is given by

$$\mathbf{v}^{P_i} \approx \sum_{r=1}^n \mathbf{v}_r^{P_i}(0) \dot{q}_r \quad (i = 1, \dots, v)$$

**\*12.5** Referring to Problem 9.9, letting  $A$  be a Newtonian reference frame, and using as generalized speeds for  $B$  in  $A$  the quantities  $\dot{q}_1, \dot{q}_2$ , and  $\dot{q}_3$ , form linearized dynamical equations governing  $q_1, q_2, q_3$ . Do this by employing the result developed in Problem 12.4.

**Result**

$$\ddot{q}_1 \approx 0$$

$$\ddot{q}_2 + 3Gm \frac{I_3 - I_1}{I_2 R^3} q_2 \approx 0$$

$$\ddot{q}_3 - 3Gm \frac{I_1 - I_2}{I_3 R^3} q_3 \approx 0$$

**12.6** Two uniform bars,  $B_1$  and  $B_2$ , each of length  $L$  and mass  $m$ , are supported by pins, as indicated in Fig. P12.6, and are attached to each other by a light, linear spring of natural length  $L/4$  and spring constant  $mg/L$ . Formulate equations that govern the angles  $q_1$  and  $q_2$  (see Fig. P12.6) when  $B_1$  and  $B_2$  are at rest, and verify that two sets of values satisfying these equations are  $q_1 = 35.55^\circ$ ,  $q_2 = 44.91^\circ$  and  $q_1 = -114.69^\circ$ ,  $q_2 = -94.77^\circ$ .

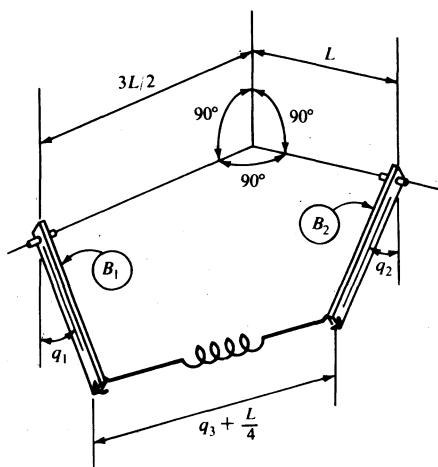


Figure P12.6

*Suggestion:* Introduce the stretch of the spring as a pseudo-generalized coordinate  $q_3$  that must satisfy the configuration constraint equation

$$\left(q_3 + \frac{L}{4}\right)^2 = L^2 \left[ (1 - s_1)^2 + \left(\frac{3}{2} - s_2\right)^2 + (c_1 - c_2)^2 \right]$$

After defining generalized speeds  $u_1$ ,  $u_2$ , and  $u_3$  ( $u_3$  is a pseudo-generalized speed) as

$$u_r \triangleq \dot{q}_r \quad (r = 1, 2, 3)$$

and differentiating the configuration constraint equation to obtain a motion constraint equation having the form of Eqs. (5.1.13), use Eqs. (6.5.4).

### Results

$$\left(\frac{q_3}{L} + \frac{1}{4}\right)s_1 + 2\frac{q_3}{L}(c_2s_1 - c_1) = 0$$

$$\left(\frac{q_3}{L} + \frac{1}{4}\right)s_2 + 2\frac{q_3}{L}\left(c_1s_2 - \frac{3}{2}c_2\right) = 0$$

**12.7** Two particles,  $P_1$  and  $P_2$ , of masses  $m_1$  and  $m_2$ , respectively, are supported by a light linkage, as indicated in Fig. P8.6, where  $\mathbf{a}_x$  and  $\mathbf{a}_y$  are unit vectors directed vertically downward and horizontally, respectively. Assuming that this system is at rest, verify that the parameters  $L_1$ ,  $L_2$ ,  $L_3$ ,  $L_4$ ,  $m_1$ , and  $m_2$  are related to the variables  $q_1$ ,  $q_2$ , and  $q_3$  as follows:

$$L_1c_1 + L_2c_2 - L_3c_3 = 0$$

$$L_1s_1 + L_2s_2 - L_3s_3 - L_4 = 0$$

$$m_1s_1 + m_2s_3 \frac{\sin(q_2 - q_1)}{\sin(q_2 - q_3)} = 0$$

**12.8** Referring to Problem 8.7, formulate three equations relating  $Q$ ,  $R$ ,  $S$ ,  $q_1$ ,  $q_2$ ,  $q_3$ , and  $m$  when all rods are at rest.

### Result

$$(-Q + R + S)c_1 - 30mgs_1 = 0$$

$$(Q - R)c_2 - 19mgs_2 = 0$$

$$Rc_3 - 8mgs_3 = 0$$

**\*12.9** Referring to Problem 8.10, formulate an equation relating  $T_B$  to  $T_E$  when  $B$ ,  $C$ ,  $C'$ ,  $D$ ,  $D'$ ,  $E$ , and  $F$  are at rest.

**Result**  $T_B + 244T_E = 0$

**12.10** Referring to Problem 3.11, and letting  $T\mathbf{N}$ ,  $T'\mathbf{N}$ , and  $t\mathbf{n}$  be the torques of couples applied to  $A$ ,  $A'$ , and  $D$ , respectively show that  $T$ ,  $T'$ , and  $t$  satisfy the equations

$$T = T' = -\frac{ta}{2d}$$

when the system is at rest.

**12.11** Figure P12.11 shows four bars, each of length  $L$ , connected by hinges and linear springs, the springs having spring constants  $k_1$  and  $k_2$  and equal natural lengths  $L/2$ . Neglecting gravitational effects, show that when this system is at rest, then

$$\frac{k_1}{k_2} = \frac{1 - \frac{1}{2}[2(1 - \cos q)]^{-1/2}}{1 - \frac{1}{2}[2(1 + \cos q)]^{-1/2}}$$

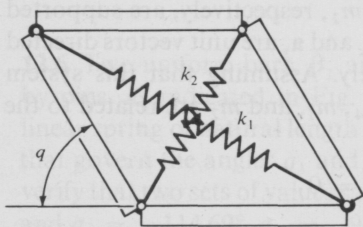


Figure P12.11

**12.12** Referring to Problem 12.2, consider the steady motion characterized by the equations

$$q_1 = \bar{q}_1 \quad \omega^A = p\mathbf{a}_1$$

where  $\bar{q}_1$  and  $p$  are constants. Under these circumstances,  $p$  is called the rate of *precession* of the gyroscope, and the motion is termed a *steady precession*. Determine the relationship among  $m, h, J_1, J_2, K_1, K_2, \bar{q}_1, s$ , and  $p$  that prevails during steady precession.

**Result**  $p^2(J_1 + K_1 - J_2 - K_2 - mh^2) \cos \bar{q}_1 + psK_1 - mgh = 0$

**12.13** In Fig. P12.13,  $A, \dots, E$  are uniform, square plates, each having a mass  $m$  and sides of length  $L$ . These plates are attached to each other and to a uniform square plate  $F$  having a mass  $5m$  and sides of length  $L$ , by means of smooth hinges.

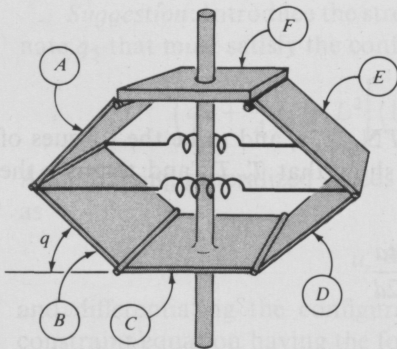


Figure P12.13

A vertical shaft, to which  $C$  is attached rigidly, passes through an opening in  $F$ , thus leaving  $F$  free to move up and down. Finally, two light, linear springs, each having a natural length  $L$  and spring constant  $k$ , connect the plates as shown.

One possible motion of this system is the following: The shaft is made to rotate with a constant angular speed  $\Omega$ , and  $q$  (see Fig. P12.13) remains constant. Show that

$$\frac{L\Omega^2}{g} = \frac{6 \cot q}{3 + 4 \cos q} \left( 4 \frac{kL}{mg} \sin q - 7 \right)$$

under these circumstances.

## PROBLEM SET 13

(Secs. 7.1–7.7)

**13.1** Referring to Problem 8.6, let  $L_1 = L_4 = 2$  m,  $L_2 = L_3 = 3$  m,  $m_1 = 4$  kg,  $m_2 = 5$  kg, and suppose that the system is released from rest when  $q_1 = 45^\circ$ . Determine the value of  $\dot{q}_1$  for the first instant at which  $q_1$  vanishes.

*Note:* It can be verified that  $q_2$  and  $q_3$  have the values  $82.281^\circ$  and  $52.719^\circ$ , respectively, when  $q_1 = 45^\circ$ , and that  $q_2 = 73.126^\circ$ ,  $q_3 = 16.874^\circ$  when  $q_1 = 0$ .

**Result**  $-1.87$  rad/s

**13.2** Letting  $q_1, \dots, q_n$  be generalized coordinates of a holonomic system in a reference frame  $N$ , show that  $H$ , the Hamiltonian of  $S$  in  $N$  (see Sec. 7.2), can be expressed as

$$H = \sum_{r=1}^n \frac{\partial \mathcal{L}}{\partial \dot{q}_r} \dot{q}_r - \mathcal{L}$$

where  $\mathcal{L}$  is the Lagrangian of  $S$  in  $N$  (see Problem 11.13), regarded as a function of  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ , and  $t$ .

*Suggestion:* Take advantage of the fact that [see Eqs. (5.5.7)–(5.5.9)]  $K_0$ ,  $K_1$ , and  $K_2$  can be written, respectively, as

$$K_0 = A \quad K_1 = \sum_{r=1}^n B_r \dot{q}_r \quad K_2 = \frac{1}{2} \sum_{r=1}^n \sum_{s=1}^n C_{rs} \dot{q}_r \dot{q}_s$$

where  $A$ ,  $B_r$ , and  $C_{rs}$  are independent of  $\dot{q}_1, \dots, \dot{q}_n$ , and  $C_{rs} = C_{sr}$  ( $r, s = 1, \dots, n$ ).

**13.3** The axis of a circular disk  $B$  of radius  $R$  (see Fig. P13.3) is fixed, and  $B$  is made to rotate about this axis with a constant angular speed  $\omega$ . A vane  $V$  is fixed on  $B$ , the equation of the center line of  $V$  being  $r = R \sin 2q$ , where  $r$  is the distance from the axis of  $B$  to  $P$ , a generic point of the center line, and  $q$  is the angle between line  $OP$  and a line  $OQ$  that is fixed on  $B$ . Finally, a particle is free to move in  $V$ .

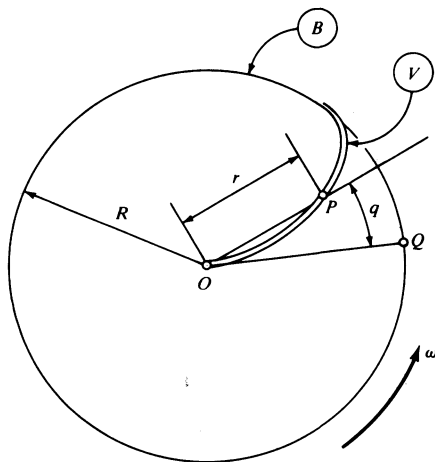


Figure P13.3

If the axis of  $B$  is vertical, and if the particle is inserted into  $V$  at  $O$  with a very small velocity (essentially zero), the particle moves toward the periphery of  $B$  and arrives there with a velocity having a magnitude  $v$ . Determine  $v$ .

**Result**  $v = 2R\omega$

**13.4** If the plane  $H$  in Problem 8.3 is perfectly smooth, the equations of motion of  $C$  possess an integral of the form

$$u_2 c_2 + \alpha u_3 s_2 = \beta$$

where  $\alpha$  is a definite constant and  $\beta$  is an arbitrary constant. Determine  $\alpha$ .

**Result**  $\alpha = 2$

**\*13.5** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, 8.21, 10.10, 11.7, and 11.8), determine  $\mathbf{H} \cdot \mathbf{a}_1$ , where  $\mathbf{H}$  is the angular momentum of the robot arm in  $E$  with respect to a point on the hub axis of  $A$ .

**Result**

$$\begin{aligned} \mathbf{H} \cdot \mathbf{a}_1 = & -m_A L_A Z_5 - L_P (m_B Z_8 + m_C Z_{12} + m_D Z_{18}) + c_1 [-m_B L_B Z_8 \\ & - m_C Z_9 Z_{12} + m_D (p_2 Z_{19} - Z_{14} Z_{18}) + (B_1 + C_1) Z_1] \\ & + s_1 [m_D (-p_2 Z_{17} + p_1 Z_{18}) + (B_3 + C_3) Z_2] + Z_{49} Z_1 \\ & + Z_{51} u_2 + Z_{53} Z_2 + A_1 u_1 \end{aligned}$$



**\*13.6** If  $\mathcal{L}$  is the kinetic potential of a holonomic system  $S$  whose first  $k$  of  $n$  generalized coordinates are ignorable (see Sec. 7.3), and if  $\alpha_r$  are the associated constant values of  $\partial\mathcal{L}/\partial\dot{q}_r$  ( $r = 1, \dots, k$ ), then the quantity  $\mathcal{R}$  defined as

$$\mathcal{R} \triangleq \mathcal{L} - \sum_{s=1}^k \alpha_s \dot{q}_s$$

and called the *Routhian* of  $S$ , may be regarded as a function of  $\rho_1, \dots, \rho_{n-k}, \dot{\rho}_1, \dots, \dot{\rho}_{n-k}$ , and  $t$ , where  $\rho_r$  is defined as

$$\rho_r \triangleq q_{k+r} \quad (r = 1, \dots, n-k)$$

Show that  $\mathcal{R}$  satisfies the equations

$$\frac{d}{dt} \frac{\partial \mathcal{R}}{\partial \dot{\rho}_r} - \frac{\partial \mathcal{R}}{\partial \rho_r} = 0 \quad (r = 1, \dots, n-k)$$

**13.7** Show without explicit use of differential equations of motion that the relationships

$$(3a^2 + b^2 s_2^2) \dot{q}_1^2 + b^2 \dot{q}_2^2 + 3abc_2 \dot{q}_1 \dot{q}_2 - 4gc_2 = \alpha$$

and

$$2(3a^2 + b^2 s_2^2) \dot{q}_1 + 3abc_2 \dot{q}_2 = \beta$$

where  $\alpha$  and  $\beta$  are constants, are integrals of the equations of motion of the triangular plate in Problem 10.5.

**13.8** Figure P13.8 shows a gyroscopic device consisting of a frame  $F$ , a torsion spring assembly  $S$ , a gimbal ring  $G$ , and a rotor  $R$ . These parts have the following inertia properties:

The point of intersection of the spin axis and the output axis is the mass center of  $R$ , and  $R$  has a moment of inertia  $J$  about the spin axis, and a moment of inertia  $I$  about any line passing through the mass center of  $R$  and perpendicular to the spin axis. The mass center of  $G$  coincides with that of  $R$ , and the spin axis, the output axis, and a line perpendicular to both of these and passing through their intersection all are principal axes of  $G$ , the corresponding moments of inertia being  $A$ ,  $B$ , and  $C$ . Finally, the mass center of  $F$  lies on the input axis, and  $F$  has a moment of inertia  $D$  about this axis.

Consider the following class of motions of the device:  $F$  is free to rotate about the input axis, which is fixed;  $G$  can rotate about the output axis, but a resisting torque of magnitude  $kq_1$  is associated with such rotations; and  $R$  is made to rotate with constant angular speed  $\Omega$  in  $G$  (by means of a motor connecting  $R$  to  $G$ ). Assuming that at time  $t = 0$  the frame  $F$  is at rest,  $q_1 = \pi/2$  rad, and  $\dot{q}_1 = 0$ , determine the value of the spring constant  $k$  such that  $\dot{q}_1$  vanishes for  $t > 0$  when  $q_1 = \pi/4$  rad.

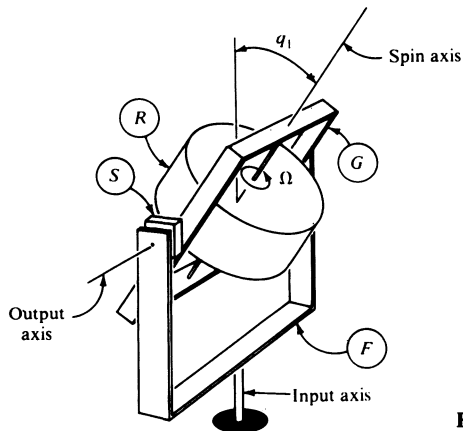


Figure P13.8

**Result** 
$$\frac{(4J\Omega/\pi)^2}{3(A + C + I + J + 2D)}$$

**13.9** Referring to the example in Sec. 7.4, suppose that the set of friction forces exerted on  $S$  by  $T$  is treated as equivalent to a couple of torque  $-cu_1\mathbf{e}_1$  (see Fig. 7.4.1), where  $c$  is a positive constant. Taking  $u_1 = u_4 = 0$  and  $u_5 = 4c(R - r)/(mr^2)$  at  $t = 0$ , show that  $u_4$  approaches a limiting value  $u_4^*$  as  $t$  approaches infinity, and plot  $u_4/u_4^*$  versus  $ct/(mr^2)$ .

**Result** Figure P13.9

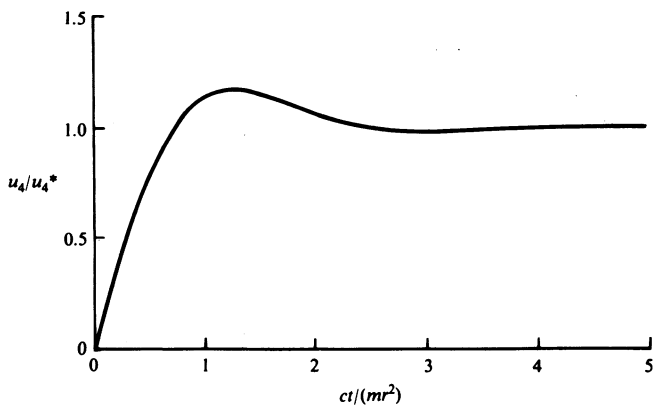


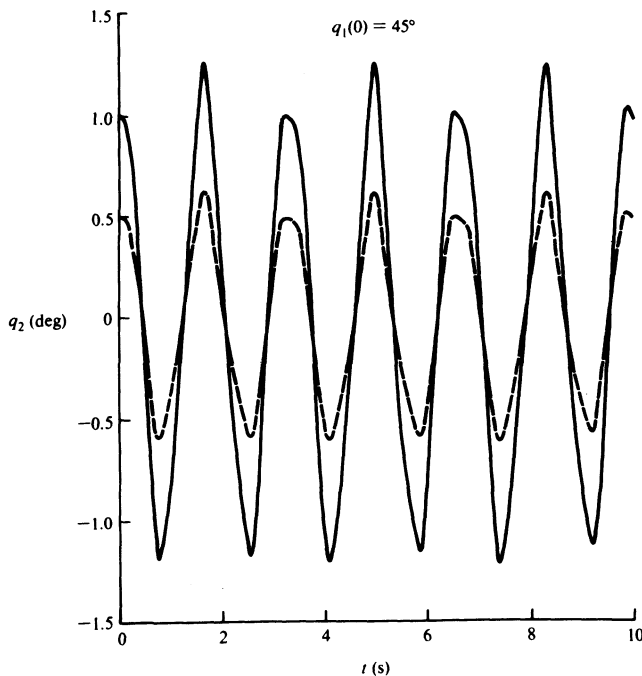
Figure P13.9

**13.10** Considering the system  $S$  introduced in Problem 12.3, and letting  $u_r \triangleq \dot{q}_r$  ( $r = 1, 2$ ), (a) formulate exact dynamical equations of  $S$ . (b) Taking  $m_A = 0.01$  kg,  $m_B = 0.1$  kg,  $L_A = 0.075$  m,  $L_B = 0.2$  m,  $A_1 = 5.0 \times 10^{-6}$  kg m<sup>2</sup>,  $B_1 = 2.5 \times 10^{-4}$  kg m<sup>2</sup>,  $B_2 = 5.0 \times 10^{-5}$  kg m<sup>2</sup>, and  $B_3 = 2.0 \times 10^{-4}$  kg m<sup>2</sup>, perform a numerical simulation of the motion of  $S$  for  $0 \leq t \leq 10$  s, using for initial conditions  $u_1(0) = u_2(0) = 0$ ,  $q_1(0) = 45^\circ$ ,  $q_2(0) = 1^\circ$ , and plot  $q_2$  versus  $t$ . Leaving all other quantities unchanged, but taking  $q_2(0) = 0.5^\circ$ , make another plot of  $q_2$  versus  $t$ , and display the two curves on the same set of axes. (c) Repeat part (b) with  $q_1(0) = 90^\circ$ .

Compare the set of two curves generated in part (b) with the set obtained in part (c), and briefly state your conclusions.

### Results

- (a)  $\dot{u}_1 = [2u_1u_2s_2c_2(B_1 - B_2) - (m_AL_A + m_BL_B)gs_1]/(A_1 + B_1c_2^2 + B_2s_2^2 + m_AL_A^2 + m_BL_B^2)$ ;  
 $\dot{u}_2 = -u_1^2s_2c_2(B_1 - B_2)/B_3$
- (b) Fig. P13.10(a)
- (c) Fig. P13.10(b)



**Figure P13.10(a)**

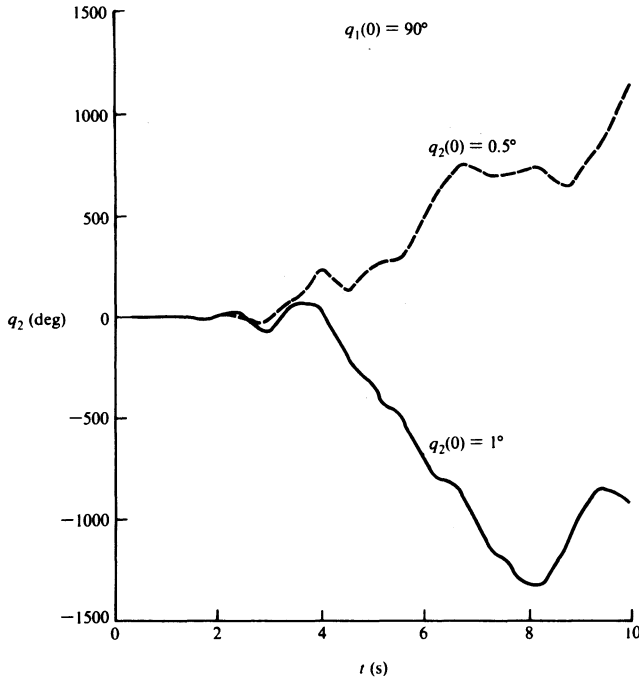


Figure P13.10(b)

**\*13.11** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, 8.21, 10.10, 11.7, 11.8, and 13.5), take  $L_A = 0.3$  m,  $L_B = 0.5$  m,  $L_P = 1.1$  m,  $p_1 = 0.2$  m,  $p_2 = 0.4$  m,  $p_3 = 0.6$  m,  $A_1 = 11$  kg m<sup>2</sup>,  $B_1 = 7$  kg m<sup>2</sup>,  $B_2 = 6$  kg m<sup>2</sup>,  $B_3 = 2$  kg m<sup>2</sup>,  $C_1 = 5$  kg m<sup>2</sup>,  $C_2 = 4$  kg m<sup>2</sup>,  $C_3 = 1$  kg m<sup>2</sup>,  $D_{11} = 2$  kg m<sup>2</sup>,  $D_{22} = 2.5$  kg m<sup>2</sup>,  $D_{33} = 1.3$  kg m<sup>2</sup>,  $D_{12} = 0.6$  kg m<sup>2</sup>,  $D_{23} = 0.75$  kg m<sup>2</sup>,  $D_{31} = -1.1$  kg m<sup>2</sup>,  $m_A = 87$  kg,  $m_B = 63$  kg,  $m_C = 42$  kg,  $m_D = 50$  kg. (a) Setting  $u_1(0) = 0.1$  rad/s,  $u_2(0) = 0.2$  rad/s,  $u_3(0) = 0.3$  m/s,  $q_1(0) = 30^\circ$ ,  $q_2(0) = 0.1$  m,  $q_3(0) = 10^\circ$ ,  $T_1^{E/A} = T_2^{A/B} = K_3^{B/C} = 0$ , determine the value of  $K + V_\gamma$  (see Problem 10.10) at  $t = 0$  and  $t = 5$  s, and the value of  $\mathbf{H} \cdot \mathbf{a}_1$  (see Problem 13.5) at  $t = 0$  and  $t = 5$  s; (b) setting  $T_2^{A/B} = 10$  N m and  $K_3^{B/C} = 20$  N, but leaving all other quantities as in (a), determine the value of  $K + V_\gamma$  at  $t = 0$  and  $t = 5$  s, and the value of  $\mathbf{H} \cdot \mathbf{a}_1$  at  $t = 0$  and  $t = 5$  s; (c) the robot arm can be brought from an initial state of rest in  $E$  to a final state of rest in  $E$  such that  $q_1$ ,  $q_2$ , and  $q_3$  have the specified values  $q_1^*$ ,  $q_2^*$ , and  $q_3^*$ , respectively, by using the following feedback control laws:

$$T_1^{E/A} = -\beta^{E/A}u_1 - \gamma^{E/A}(q_3 - q_3^*)$$

$$T_2^{A/B} = -\beta^{A/B}u_2 - \gamma^{A/B}(q_1 - q_1^*) + g[(m_B L_B + m_C Z_9 + m_D Z_{14})c_1 - m_D p_1 s_1]$$

$$K_3^{B/C} = -\beta^{B/C}u_3 - \gamma^{B/C}(q_2 - q_2^*) + g(m_C + m_D)s_1$$

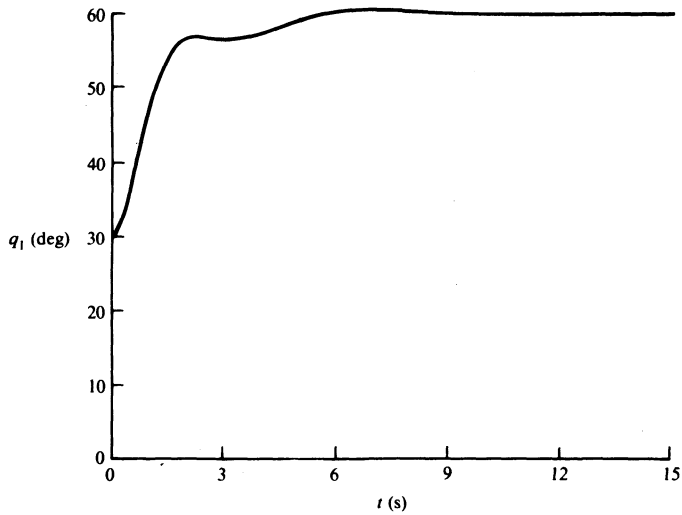


Figure P13.11(a)

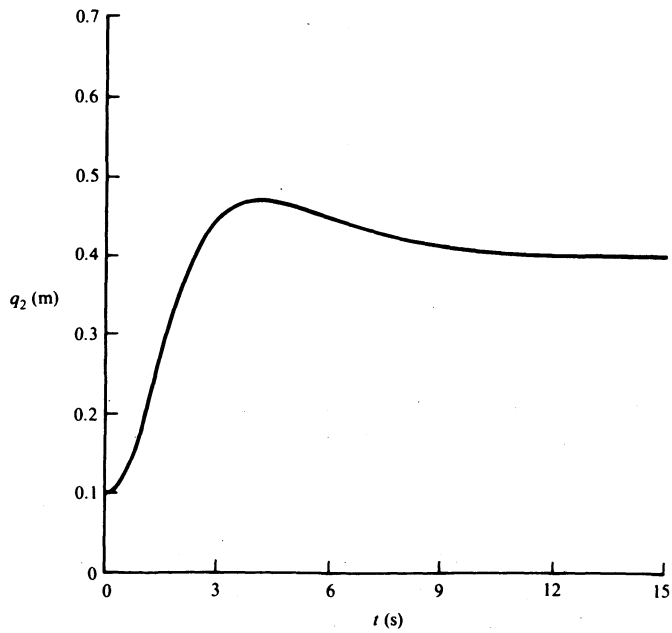


Figure P13.11(b)

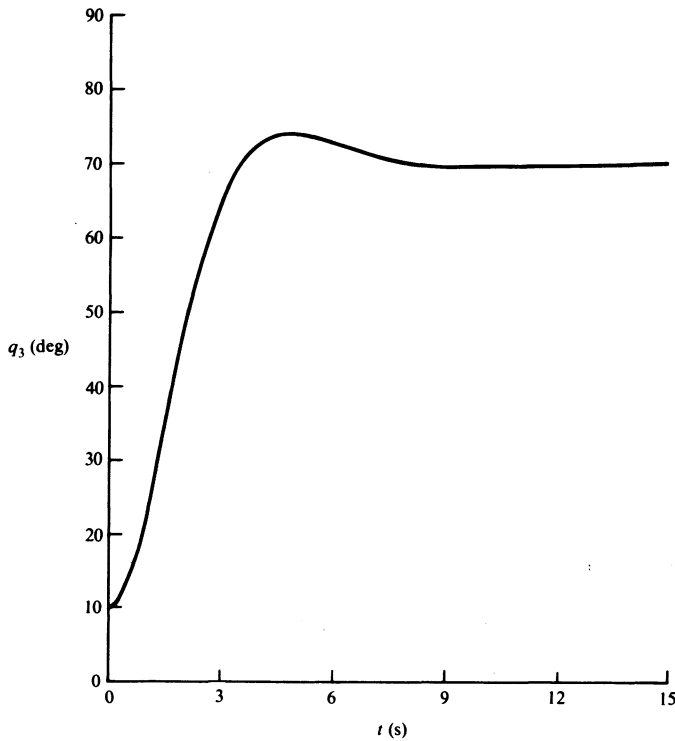


Figure P13.11(c)

Here,  $\beta^{E/A}$ ,  $\gamma^{E/A}$ ,  $\beta^{A/B}$ ,  $\gamma^{A/B}$ ,  $\beta^{B/C}$ , and  $\gamma^{B/C}$  are constant “gains,” and the terms involving  $g$  in the expressions for  $T_2^{A/B}$  and  $K_3^{B/C}$  serve to counteract the effects of gravity. Using these control laws, the parameter values employed previously, and  $u_1(0) = u_2(0) = u_3(0) = 0$ ,  $q_1(0) = 30^\circ$ ,  $q_2(0) = 0.1$  m,  $q_3(0) = 10^\circ$ ,  $q_1^* = 60^\circ$ ,  $q_2^* = 0.4$  m,  $q_3^* = 70^\circ$ ,  $\beta^{E/A} = 464$  N m s,  $\gamma^{E/A} = 306$  N m,  $\beta^{A/B} = 216$  N m s,  $\gamma^{A/B} = 285$  N m,  $\beta^{B/C} = 169$  N s,  $\gamma^{B/C} = 56$  N, plot the values of  $q_1$ ,  $q_2$ , and  $q_3$  from  $t = 0$  to  $t = 15$  s.

### Results

(a)  $K + V_\gamma$ : 666.00 N m, 666.00 N m

$H_1$ : 52.643 N m s, 52.643 N m s

(b)  $K + V_\gamma$ : 666.00 N m, 3116.6 N m

$H_1$ : 52.643 N m s, 52.643 N m s

(c) Figures P13.11(a), P13.11(b), P13.11(c)

**13.12** Referring to the example in Sec. 7.6, replace the set of forces exerted on the left half of  $C$  by the right half with a couple together with a force  $\sigma_1 \mathbf{n}_1 + \sigma_2 \mathbf{n}_2$ , applying this force at the right end of the left half of  $C$ . For the values of  $m_A$ ,  $m_B$ ,  $m_C$ ,  $L$ ,  $q_1(0)$ , and  $u_1(0)$  used in the example in Sec. 7.6, show that  $\sigma_1 = -(g/16) \tan q_1$ , and determine  $\sigma_1$  for  $t = 5$  s.

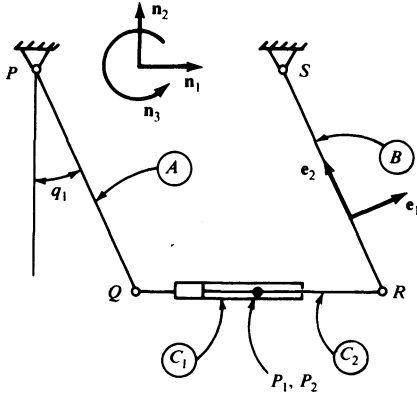


Figure P13.12

*Suggestions:* Regard  $C$  as composed of two identical bars,  $C_1$  and  $C_2$ , each of length  $L$  and mass  $m_C/2$ , which can slide relative to each other as indicated in Fig. P13.12, and let  $P_1$  and  $P_2$  be the points of  $C_1$  and  $C_2$ , respectively, corresponding to the midpoint of  $C$ . Note that during the motion of interest the velocity of  $P_2$  in  $C_1$  and the angular velocities of  $C_1$  and  $C_2$  can be expressed as

$${}^{C_1}\mathbf{v}^{P_2} = v\mathbf{n}_1 \quad \boldsymbol{\omega}^{C_1} = \boldsymbol{\omega}^{C_2} = \gamma\mathbf{n}_3$$

where  $v$  and  $\gamma$  are functions of  $q_1$  and  $u_1$ . Introduce  $u_2$  as  $u_2 \triangleq \boldsymbol{\omega}^B \cdot \mathbf{n}_3$ , and verify that the velocities of  $P_1$  and  $P_2$  are given by

$$\mathbf{v}^{P_1} = 2Lu_1\mathbf{e}_1 + L(u_2 - u_1)s_1\mathbf{n}_2$$

and

$$\mathbf{v}^{P_2} = 2Lu_2\mathbf{e}_1 - L(u_2 - u_1)s_1\mathbf{n}_2$$

**Result**  $\sigma_1 = -0.173 \text{ N}$

**13.13** The system described in Problem 8.7 is at rest with  $q_1 = q_2 = q_3 = 45^\circ$ . Letting  $B$  designate the third bar from the left in the middle row of bars (the row in which the bars make an angle  $q_2$  with the vertical), determine the magnitude of the reaction of  $B$  on the pin supporting the upper end of  $B$ .

*Suggestion:* Regard the upper end of  $B$  as disconnected from the pin that supports this end, and note that the six bars shown in Fig. P13.13(a) are movable

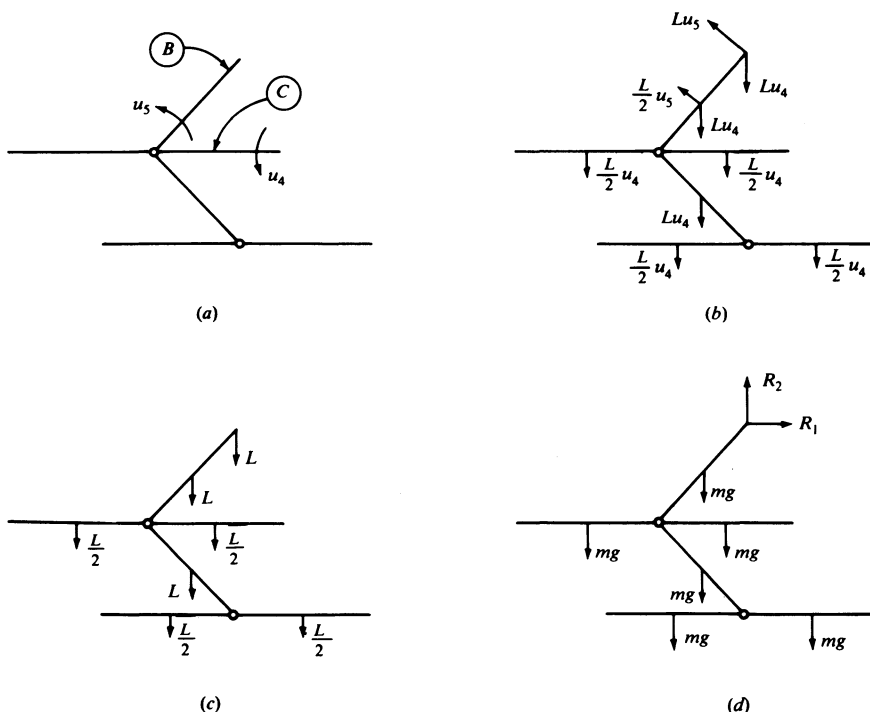


Figure P13.13

under these circumstances, whereas the rest of the bars are not movable so long as  $q_1$ ,  $q_2$ , and  $q_3$  have fixed values. Introduce generalized speeds  $u_4$  and  $u_5$  such that bars  $B$  and  $C$  have counterclockwise angular velocities, as indicated in Fig. P13.13(a). Record the velocities of the mass centers of the movable bars and the velocity of the upper end of  $B$  as in Fig. P13.13(b), and use this sketch to depict partial velocities associated with  $u_4$  as in Fig. P13.13(c). Record the gravitational forces acting on the six bars, as well as the horizontal and vertical reaction force components at the upper end of  $B$  as in Fig. P13.13(d). Refer to Figs. P13.13(c) and P13.13(d) to perform by inspection the dot-multiplications required to form the generalized active force  $F_4$ , thus verifying that  $F_4 = L(-R_2 + 4mg)$ . Make a sketch similar to Fig. P13.13(c) to depict the partial velocities associated with  $u_5$ , and use this sketch in conjunction with Fig. P13.13(d) to form an expression for the generalized active force  $F_5$ . Finally, appeal to Eqs. (6.5.2) to find  $R_1$  and  $R_2$ .

**Result**  $\sqrt{113mg}/2$

**\*13.14** Referring to the example in Sec. 6.6, verify that there exist values of  $\theta$  ( $0 < \theta < \pi/2$ ) such that the steady motion there considered cannot occur. For



$0 \leq h/R \leq 2$ , show that these values of  $\theta$  correspond to the shaded region of Fig. P13.14.

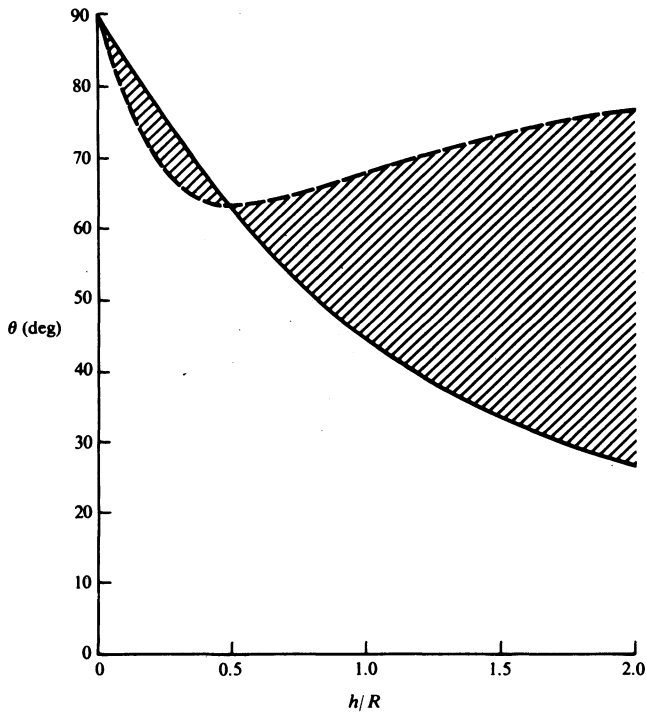


Figure P13.14

**13.15** Taking  $L\Omega^2/g = 3$ , determine two pairs of constant values of  $q_1$  and  $q_2$  such that the equations of motion of the system considered in the example in Sec. 6.7 are satisfied, and sketch the system in the configuration associated with each pair of values.

**Results**  $q_1 = 56.18^\circ$ ,  $q_2 = 226.30^\circ$ ;  $q_1 = 74.25^\circ$ ,  $q_2 = 78.34^\circ$ ; Fig. P13.15

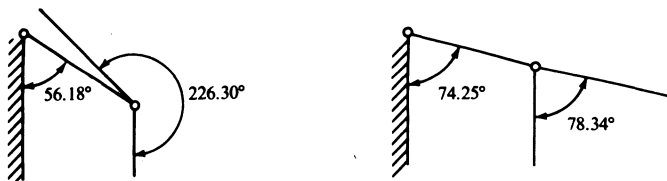


Figure P13.15

# PROBLEM SET 14

(Secs. 7.8–7.10)

**14.1** A rigid body  $B$  forms part of a simple nonholonomic system  $S$  possessing  $n - m$  degrees of freedom in a Newtonian reference frame  $N$ . Letting  $\mathbf{L}$  and  $\mathbf{H}$  denote, respectively, the linear momentum of  $B$  in  $N$  and the central angular momentum of  $B$  in  $N$ , show that the contribution of  $B$  to the generalized momenta  $p_1, \dots, p_{n-m}$  can be expressed as

$$(p_r)_B = \tilde{\omega}_r \cdot \mathbf{H} + \tilde{v}_r^* \cdot \mathbf{L} \quad (r = 1, \dots, n - m)$$

where  $\tilde{\omega}_r$  and  $\tilde{v}_r^*$  are, respectively, the  $r$ th nonholonomic partial angular velocity of  $B$  in  $N$  and the  $r$ th nonholonomic partial velocity of the mass center of  $B$  in  $N$ .

**14.2** A rigid body  $B$  is a part of a simple nonholonomic system  $S$  possessing  $n - m$  degrees of freedom in a Newtonian reference frame  $N$ , and a subset  $\sigma$  of the contact forces acting on particles of  $S$  consists of forces applied to  $B$  and forming a couple of torque  $\mathbf{T}$ . Show that  $(I_r)_\sigma$ , the contribution of the forces of  $\sigma$  to the generalized impulse  $I_r$ , is given by

$$(I_r)_\sigma = \tilde{\omega}_r^B(t_1) \cdot \int_{t_1}^{t_2} \mathbf{T} dt \quad (r = 1, \dots, n - m)$$

where  $\tilde{\omega}_r^B(t_1)$  is the value at time  $t_1$  of the  $r$ th nonholonomic partial angular velocity of  $B$  in  $N$ , and  $t_2$  differs so little from  $t_1$  that the configuration of  $S$  in  $N$  does not change significantly during the time interval beginning at  $t_1$  and ending at  $t_2$ .

**14.3** Referring to Problem 4.11, let  $P$  be a horizontal plane, and let  $A$  be a Newtonian reference frame. Show that when  $C_1$  and  $C_2$  roll on  $P$ ,  $S^*$  moves on a circle, and determine the radius of the circle when the system is set into motion by a blow applied to  $S$  along a line that intersects the axis of  $S$  at a distance  $s$  from  $S^*$ . Assume that no slipping occurs, and express the result in terms of the radius  $R$  and length  $L$  shown in Fig. P4.11, as well as the mass  $m_S$  of  $S$ , the mass  $m_C$  of each of  $C_1$  and  $C_2$ , the central transverse moment of inertia  $J$  of  $S$ , and the axial moment of inertia  $I$  of each of  $C_1$  and  $C_2$ , letting  $C_1$  and  $C_2$  each have a transverse central moment of inertia  $I/2$ .

**Result** 
$$\frac{I + J + 2(m_C + I/R^2)L^2}{s[m_S + 2(m_C + I/R^2)]}$$

**14.4** Four identical uniform rods, each of length  $2L$ , are connected by smooth pins so as to form a square, and are resting on a smooth, horizontal surface when

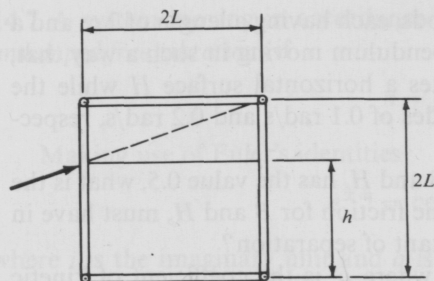


Figure P14.4

one of the rods is struck, the line of action of the blow passing through one corner of the square, as shown in Fig. P14.4. Determine the distance  $h$  such that the system moves as if it were a rigid body subsequent to being struck.

**Result**  $h = 4L/3$

**14.5** In Fig. P14.5,  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$  are mutually perpendicular unit vectors fixed relative to a horizontal plane  $H$  in such a way that  $\mathbf{n}_2$  is perpendicular to  $H$ .  $S$  is a thin, uniform spherical shell that, at a certain instant, strikes  $H$  at a point  $A$  of  $H$ , the velocity  $\mathbf{v}^*$  of the center of  $S$  and the angular velocity  $\boldsymbol{\omega}$  of  $S$  at this instant being given by

$$\mathbf{v}^* = 10(-\mathbf{n}_2 + \mathbf{n}_3) \quad \text{m/s}$$

$$\boldsymbol{\omega} = 100(\mathbf{n}_1 + 2\mathbf{n}_2 + 5\mathbf{n}_3) \quad \text{rad/s}$$

Thereafter,  $S$  bounces from  $A$  to  $B$ , from  $B$  to  $C$ , and so forth.

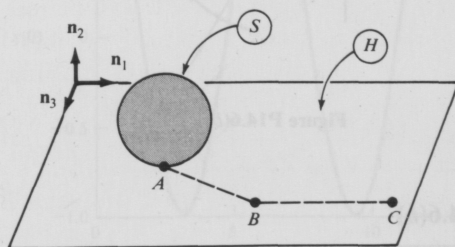


Figure P14.5

Letting the coefficient of restitution for  $S$  and  $H$  have the value 0.5, taking the coefficients of static friction and kinetic friction equal to 0.25 and 0.20, respectively, and setting  $b = 0.03$  m, determine the distance from  $A$  to  $C$ .

**Result** 13.90 m

**14.6** In Fig. P14.6(a),  $A$  and  $B$  are uniform rods, each having a length of 2 m and a mass of 3 kg. The two rods form a double pendulum moving in such a way that, at a certain instant, the free end of  $B$  strikes a horizontal surface  $H$  while the angular velocities of  $A$  and  $B$  have magnitudes of 0.1 rad/s and 0.2 rad/s, respectively, and are directed as shown.

(a) If  $e$ , the coefficient of restitution for  $B$  and  $H$ , has the value 0.5, what is the minimum value that  $\mu$ , the coefficient of static friction for  $B$  and  $H$ , must have in order for  $B$  not to be sliding on  $H$  at the instant of separation?

(b) If  $e = 0.5$ ,  $\mu = 0.25$ , and  $\mu' = 0.20$ , where  $\mu'$  is the coefficient of kinetic friction for  $B$  and  $H$ , what are the angular velocities of  $A$  and  $B$  immediately after impact? Draw a sketch of the system, showing the angular velocities of  $A$  and  $B$ .

(c) For each of the sets of values of  $e$ ,  $\mu$ , and  $\mu'$  in Table P14.6(a), determine whether or not slipping is taking place at the instant of separation, whether the kinetic energy increases or decreases during the collision, and the amount of kinetic energy change.

**Results** (a) 0.431 (b) Fig. P14.6(b) (c) Table P14.6(b)

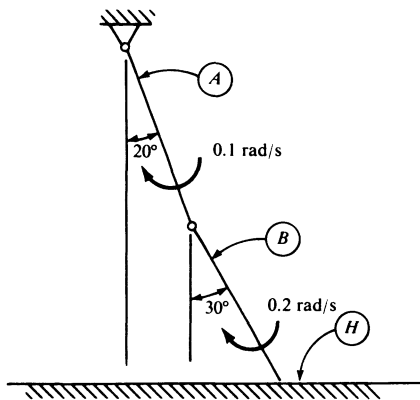


Figure P14.6(a)

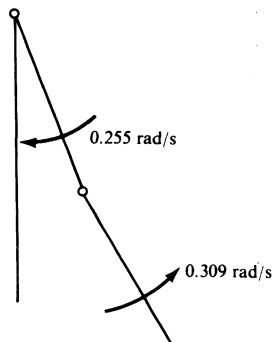


Figure P14.6(b)

Table P14.6(a)

$e$	$\mu$	$\mu'$
0.5	0.25	0.20
0.5	0.50	0.40
0.3	0.50	0.40
0.7	0.51	0.50

Table P14.6(b)

$e$	$\mu$	$\mu'$	Slipping	Kinetic energy change
0.5	0.25	0.20	Yes	0.03 N m decrease
0.5	0.50	0.40	No	0.16 N m increase
0.3	0.50	0.40	No	0.12 N m decrease
0.7	0.51	0.50	Yes	0.49 N m increase

**14.7** A system moving in accordance with Eq. (7.10.1) is said to be performing *undamped free vibrations* if

$$n = f(t) = 0$$

Making use of Euler's identities

$$e^{\pm iq} = \cos q \pm i \sin q$$

where  $i$  is the imaginary unit and  $q$  is any real quantity, verify that under these circumstances  $x$  can be expressed as

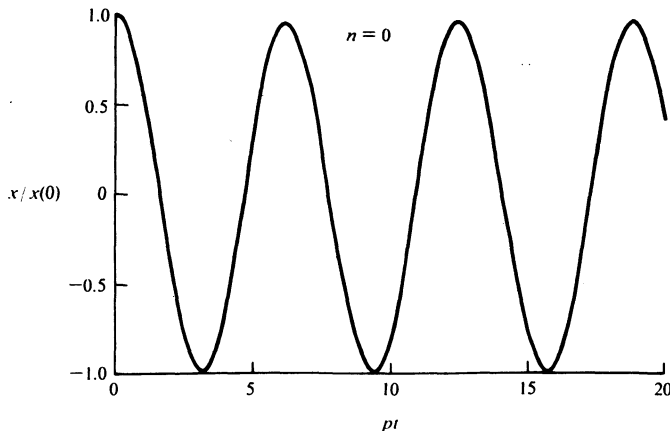
$$x = x(0) \cos pt + \frac{\dot{x}(0)}{p} \sin pt$$

Show that  $T$ , called the *period* of the motion and defined as the shortest time between two instants at which  $x$  attains stationary values (values such that  $\dot{x} = 0$ ) of the same sign, is given by

$$T = \frac{2\pi}{p}$$

Taking  $\dot{x}(0) = 0$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 20$ .

**Result** Figure P14.7



**Figure P14.7**

**14.8** A system moving in accordance with Eq. (7.10.1) is said to be performing *underdamped free vibrations* if

$$0 < n < 1 \quad f(t) = 0$$

Show that during such vibrations

$$x = \left\{ x(0) \cos[p(1 - n^2)^{1/2}t] + \frac{\dot{x}(0) + np x(0)}{p(1 - n^2)^{1/2}} \sin[p(1 - n^2)^{1/2}t] \right\} e^{-npt}$$

and verify that the period  $T$ , defined as in Problem 14.7, is given by

$$T = \frac{2\pi}{p(1 - n^2)^{1/2}}$$

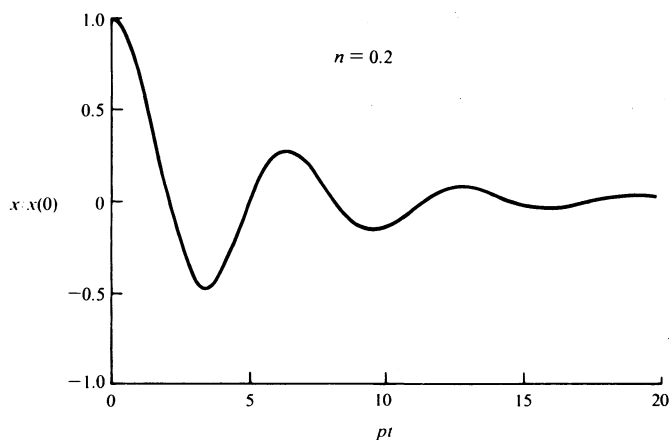


Figure P14.8(a)

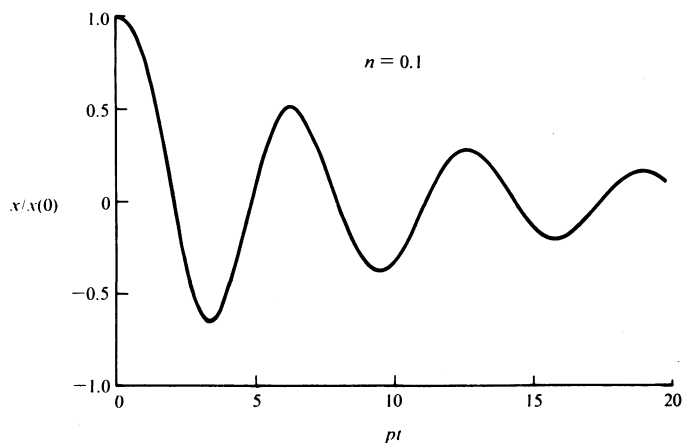


Figure P14.8(b)

Show that  $\delta$ , the *logarithmic decrement*, defined as the natural logarithm of the ratio of  $x(t)$  to  $x(t + T)$ , can be expressed as

$$\delta = npT$$

Taking  $\dot{x}(0) = 0$  and  $n = 0.2$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 20$ . Repeat with  $n = 0.1$ .

**Results** Figures P14.8(a), P14.8(b)

**14.9** A system moving in accordance with Eq. (7.10.1) is said to be performing *critically damped free vibrations* if

$$n = 1 \quad f(t) = 0$$

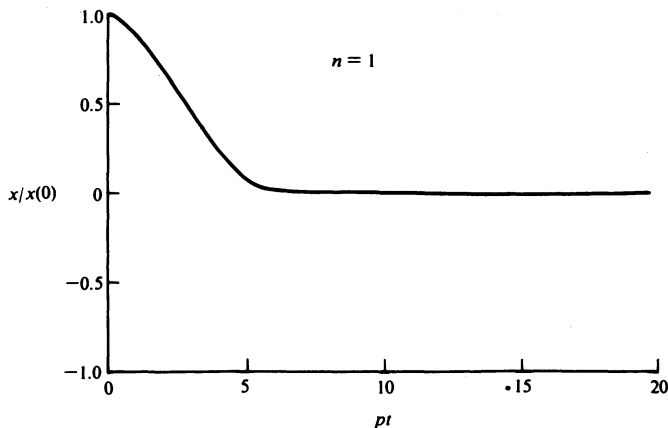
Referring to Problem 14.8, use a limiting process to verify that

$$x = \{[\dot{x}(0) + px(0)]t + x(0)\}e^{-pt}$$

and show that there exists at most one value of  $t$  such that  $x$  has a stationary value.

Taking  $x(0) = 0$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 20$ .

**Result** Figure P14.9



**Figure P14.9**

**14.10** A system moving in accordance with Eq. (7.10.1) is said to be performing *overdamped free vibrations* if

$$n > 0 \quad f(t) = 0$$

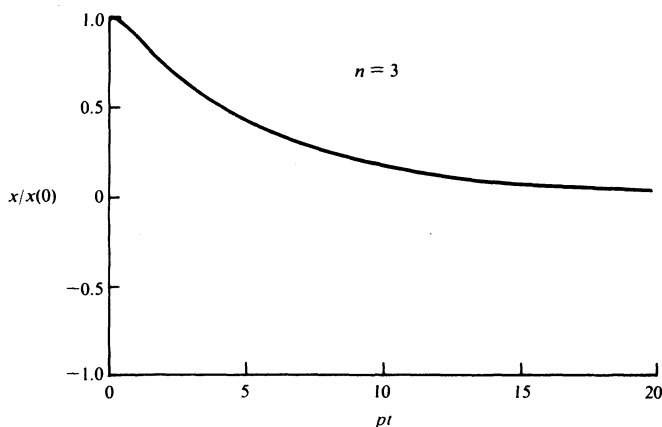
After verifying that  $x$  is given by

$$x = \frac{1}{a_1 - a_2} \{ [\dot{x}(0) - a_2 x(0)] e^{a_1 t} - [\dot{x}(0) - a_1 x(0)] e^{a_2 t} \}$$

where  $a_1$  and  $a_2$  are *real* quantities given by Eqs. (7.10.3), show that, once again (see Problem 14.9), there exists at most one value of  $t$  such that  $x$  has a stationary value.

Taking  $\dot{x}(0) = 0$  and  $n = 3$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 20$ .

**Result** Figure P14.10



**Figure P14.10**

**14.11** A system moving in accordance with Eq. (7.10.1) is said to be performing *damped harmonically forced vibrations* if

$$n > 0 \quad f(t) = b \sin(\omega t + \phi)$$

where  $b$ ,  $\omega$ , and  $\phi$  are constants called, respectively, the *amplitude*, the *circular frequency*, and the *phase angle* of the forcing function.

Show that  $x$  can be written as

$$x = x_S + x_T$$

where  $x_S$  and  $x_T$ , respectively called the *steady-state response* and the *transient response*, are given by

$$x_S = \frac{b(\beta_1 \sin \omega t + \beta_2 \cos \omega t)}{(p^2 - \omega^2)^2 + (2np\omega)^2}$$

and

$$x_T = \frac{\alpha_1 e^{a_1 t} - \alpha_2 e^{a_2 t}}{2p(n^2 - 1)^{1/2}}$$



with  $\alpha_i$  and  $\beta_i$  ( $i = 1, 2$ ) defined as

$$\alpha_1 \triangleq \dot{x}(0) - a_2 x(0) + \frac{b}{a_1^2 + \omega^2} (a_1 \sin \phi + \omega \cos \phi)$$

$$\alpha_2 \triangleq \dot{x}(0) - a_1 x(0) + \frac{b}{a_2^2 + \omega^2} (a_2 \sin \phi + \omega \cos \phi)$$

$$\beta_1 \triangleq 2np\omega \sin \phi + (p^2 - \omega^2) \cos \phi$$

$$\beta_2 \triangleq (p^2 - \omega^2) \sin \phi - 2np\omega \cos \phi$$

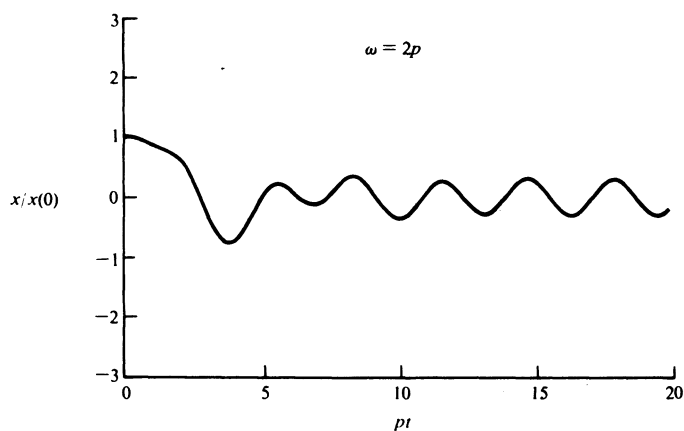


Figure P14.11(a)

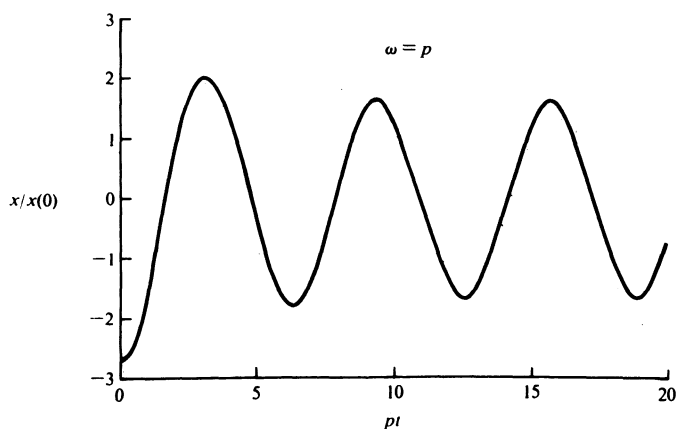


Figure P14.11(b)

and  $a_i$  ( $i = 1, 2$ ) given by Eqs. (7.10.3). Verify that

$$\lim_{t \rightarrow \infty} x_T = 0$$

and that the absolute value of the maximum value of  $x_s$ , called the *steady-state amplitude*, is given by

$$|(x_s)_{\max}| = |b|[(p^2 - \omega^2)^2 + (2np\omega)^2]^{-1/2}$$

Taking  $\dot{x}(0) = 0$ ,  $n = 0.3$ ,  $b = x(0)p^2$ ,  $\phi = 0$ , and  $\omega = 2p$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 20$ . Repeat with  $\omega = p$ .

**Results** Figures P14.11(a), P14.11(b)

**14.12** A system moving in accordance with Eq. (7.10.1) is said to be performing *undamped harmonically forced vibrations* if

$$n = 0 \quad f(t) = b \sin(\omega t + \phi)$$

where  $b$ ,  $\omega$ , and  $\phi$  are constants (see Problem 14.11).

After verifying that, so long as  $\omega$  differs from  $p$ ,  $x$  is given by

$$x = x(0) \cos pt + \frac{\dot{x}(0)}{p} \sin pt + \frac{b}{p^2 - \omega^2} \left[ \left( \sin \omega t - \frac{\omega}{p} \sin pt \right) \cos \phi + (\cos \omega t - \cos pt) \sin \phi \right]$$

show by means of a limiting process that *resonance*, that is, the response obtained when  $\omega = p$ , is characterized by

$$x = x(0) \cos pt + \left[ \frac{\dot{x}(0)}{p} + \frac{b \cos \phi}{2p^2} \right] \sin pt - \frac{bt}{2p} \cos(pt + \phi)$$

Taking  $\dot{x}(0) = 0$ ,  $b = x(0)p^2$ ,  $\omega = p$ , and  $\phi = 0$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 150$ .

When the behavior of a system is characterized by a curve such as the one in Fig. P14.12(b), *beats* are said to be taking place. Show that setting  $\dot{x}(0) = 0$ ,  $\phi = \pi/2$ , and  $b = x(0)p^2$  leads to

$$x = x(0) \cos pt - \frac{2x(0)}{1 - \omega^2/p^2} \sin \left[ \left( \frac{p + \omega}{2} \right) t \right] \sin \left[ \left( \frac{p - \omega}{2} \right) t \right]$$

Taking  $\omega = 0.9$ , plot  $x/x(0)$  versus  $pt$  for  $0 \leq pt \leq 150$  to verify that the above equation describes beats under these circumstances.

**Results** Figures P14.12(a), P14.12(b)

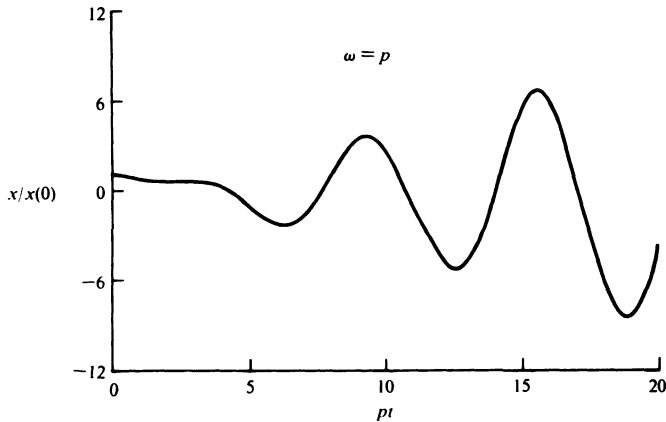


Figure P14.12(a)

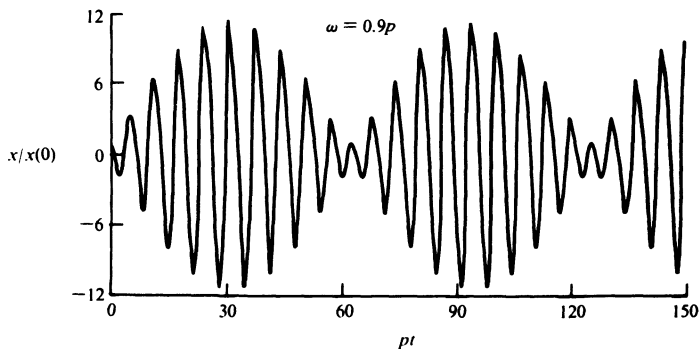


Figure P14.12(b)

**14.13** A variable-speed motor is supported by a horizontal floor. When in operation, it performs vertical vibrations giving rise to the following observations.

When the rotor is turning at a constant angular speed, the amplitude of the vertical displacement of the motor casing from the casing's rest position is greatest if the angular speed is equal to 1000 rpm; the greatest amplitude has a value of 0.011 m. Once the motor has been turned off, so that the rotor remains at rest relative to the stator, vibrations with decaying amplitude take place with a frequency of 16 Hz.

Determine at what constant angular speeds below 3000 rpm the motor may not be operated if the amplitude of the vertical displacement is not to exceed 0.005 m. For operation in the permitted range of angular speeds, find the maximum value of the ratio of  $F$  to the weight of the motor,  $F$  being the magnitude of the force transmitted to the floor by the motor casing.

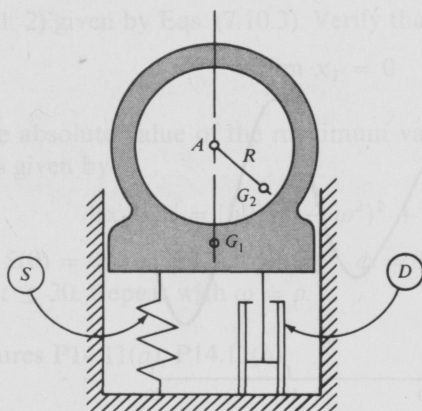


Figure P14.13(a)

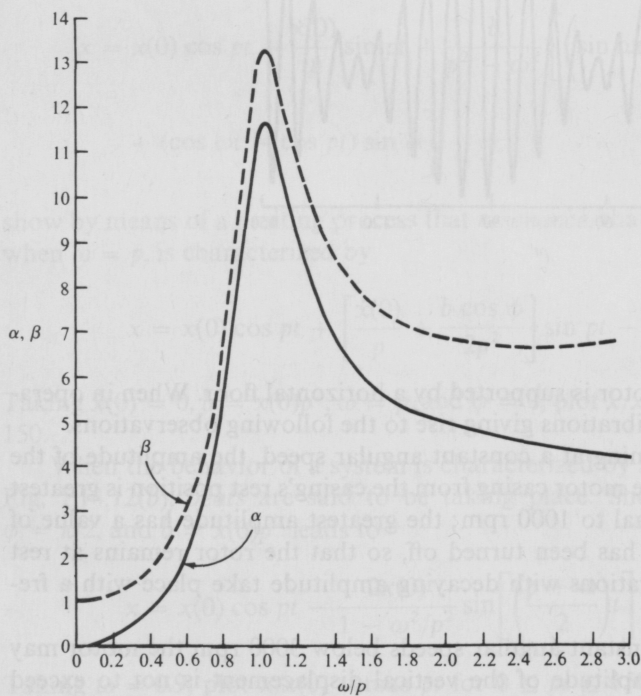


Figure P14.13(b)

**Suggestion:** Regard the motor as consisting of a rigid stator that carries a rotor whose mass center does not lie on the axis of rotation of the rotor, and let the motor be supported by a linear spring  $S$  and linear damper  $D$ , as indicated in Fig. P14.13(a), where  $G_1$  designates the mass center of the stator,  $R$  is the distance from  $A$ , the axis of rotation of the rotor, to  $G_2$ , the mass center of the rotor. Let  $h$  be the static displacement of  $A$  from the position  $A$  occupies when  $S$  is unstretched, and let  $x$  be the displacement of  $A$  from the static equilibrium position of  $A$  to the position of  $A$  at any time. Verify that  $x$  is governed by the equation

$$\ddot{x} + 2np\dot{x} + p^2x = R\left(\frac{M'}{M}\right)\omega^2 \cos \omega t$$

where  $M$  and  $M'$  are the masses of the entire motor and of the rotor, respectively,  $\omega$  is the (constant) angular speed of the rotor,  $p^2 = g/h$ , and  $2np = C/M$ ,  $C$  being the damping constant associated with  $D$ . Plot the *displacement amplification factor*  $\alpha$ , defined as  $\alpha \triangleq |x_{\max}|/h$ , and the *force amplification factor*  $\beta$ , defined as  $\beta \triangleq F_{\max}/(Mg)$ , as functions of  $\omega/p$  for  $0 \leq \omega/p \leq 3$  [see Fig. P14.13(b)].

**Results**  $775 < \omega < 1731$  rpm;  $[F/(Mg)]_{\max} = 7.1$

**\*14.14** Referring to Problem 3.15 (see also Problems 4.17, 8.14, 8.20, 8.21, 10.10, 11.7, 11.8, 13.5, and 13.11), (a) let  $I$  denote the moment of inertia of  $A$ ,  $B$ ,  $C$ , and  $D$  with respect to the hub axis of  $A$ , let  $J$  be the moment of inertia of  $B$ ,  $C$ , and  $D$  with respect to the joint axis at  $P$ , and take  $M = m_C + m_D$ . Show that (see Problem 11.7)  $I = -X_{11}$ ,  $J = -X_{22}$ , and  $M = -X_{33}$ . (b) For the parameter values and values of  $q_1(0)$ ,  $q_2(0)$ , and  $q_3(0)$  used in Problem 13.11, determine  $I$ ,  $J$ , and  $M$ . (c) Taking  $T_1^{E/A}$  as in Problem 13.11 with  $q_3^* = 0$ , and permitting only  $q_3$  to vary, show that the equation of motion governing  $q_3$ , linearized in  $q_3$ , can be written

$$\ddot{x} + 2np\dot{x} + p^2x = 0$$

where  $x = q_3$ ,  $2np = \beta^{E/A}/I$ , and  $p^2 = \gamma^{E/A}/I$ ; proceeding similarly in connection with  $q_1$  and  $q_2$ , show that this equation applies also when  $x = q_1$ ,  $2np = \beta^{A/B}/J$ , and  $p^2 = \gamma^{A/B}/J$ , in the case of  $q_1$ , and when  $x = q_2$ ,  $2np = \beta^{B/C}/M$ , and  $p^2 = \gamma^{B/C}/M$ , in the case of  $q_2$ . (d) Referring to Problem 14.8, show that

$$n = \left[ 1 + \left( \frac{2\pi}{\delta} \right)^2 \right]^{-1/2}$$

and determine  $\delta$  and  $n$  such that  $x(t + T)/x(t) = 1/100$ . (e) Taking  $T = 10$  s, determine  $\beta^{E/A}$ ,  $\gamma^{E/A}$ ,  $\beta^{A/B}$ ,  $\gamma^{A/B}$ ,  $\beta^{B/C}$ , and  $\gamma^{B/C}$ , each to the nearest whole number. (f) Using the parameters and initial conditions employed in part (c) of Problem 13.11, but assigning to  $\beta^{E/A}$ ,  $\gamma^{E/A}$ ,  $\beta^{A/B}$ ,  $\gamma^{A/B}$ ,  $\beta^{B/C}$ , and  $\gamma^{B/C}$  the values found in part (e) of the present problem, plot the values of  $q_1$ ,  $q_2$ , and  $q_3$  from  $t = 0$  to  $t = 15$  s. (g) Recompute, to the nearest whole number, the values of  $\beta^{A/B}$ ,  $\gamma^{A/B}$ , and  $\beta^{B/C}$  with  $T = 5$  s. Compare the values of  $\beta^{E/A}$ ,  $\gamma^{E/A}$ ,  $\beta^{A/B}$ ,  $\gamma^{A/B}$ ,  $\beta^{B/C}$ , and  $\gamma^{B/C}$  now in hand with their counterparts in part (c) of Problem 13.11. Compare Figs. P13.11(a)–P13.11(c) with Figs. P14.14(a)–P14.14(c). What do you conclude?

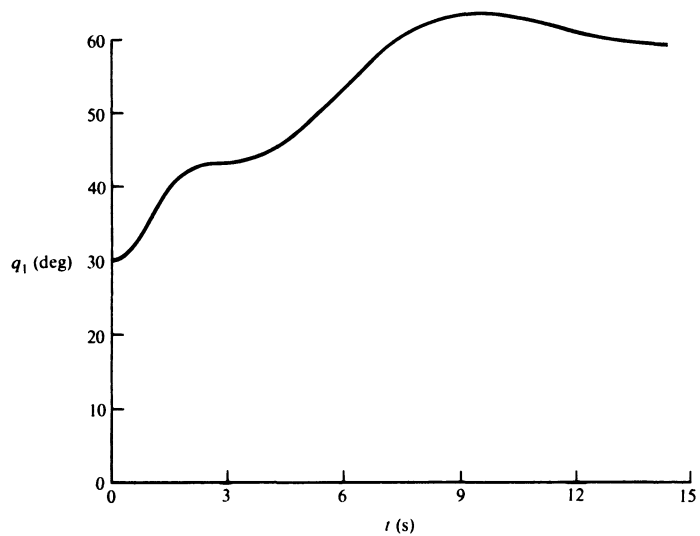


Figure P14.14(a)

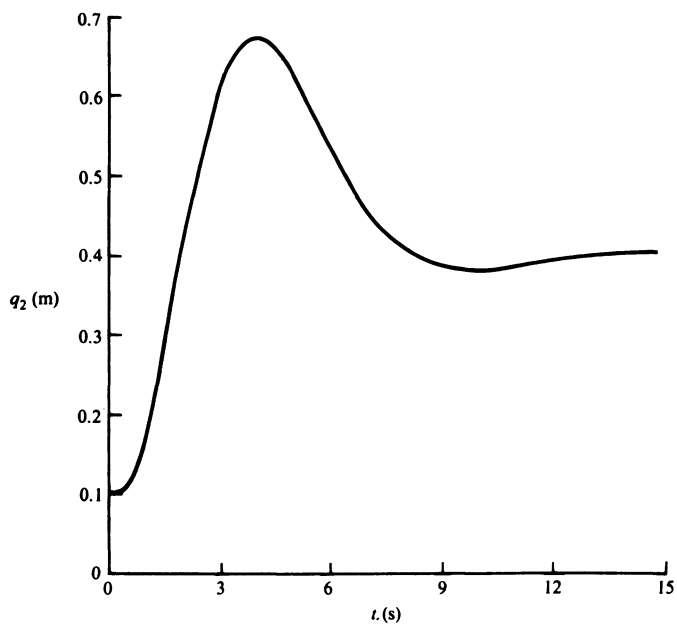


Figure P14.14(b)

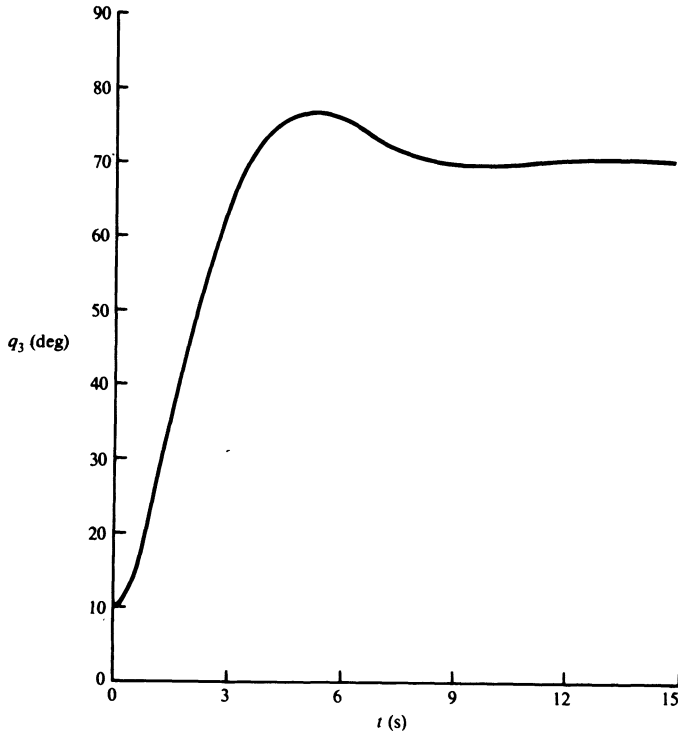


Figure P14.14(c)

- Results** (b)  $I = 503.61 \text{ kg m}^2$      $J = 117.37 \text{ kg m}^2$      $M = 92 \text{ kg}$   
 (d)  $\delta = 4.60517$      $n = 0.591155$   
 (e)  $\beta^{E/A} = 464 \text{ N m s}$      $\gamma^{E/A} = 306 \text{ N m}$      $\beta^{A/B} = 108 \text{ N m s}$   
 $\gamma^{A/B} = 71 \text{ N m}$      $\beta^{B/C} = 85 \text{ N s}$      $\gamma^{B/C} = 56 \text{ N}$   
 (f) Figures P14.14(a), P14.14(b), P14.14(c)  
 (g)  $\beta^{A/B} = 216 \text{ N m s}$      $\gamma^{A/B} = 285 \text{ N m}$      $\beta^{B/C} = 169 \text{ N s}$

**14.15** When a force  $\mu kL \sin(\varepsilon \omega t) \mathbf{n}_1$  is applied to the particle  $P_1$  of the system considered in the example in Sec. 7.10, the differential equations governing the motion of  $S$  are Eqs. (7.10.51), with  $M$  and  $K$  as given in Eqs. (7.10.59) and (7.10.61), respectively, and

$$\mathbf{F} = \mu kL [\sin(\varepsilon \omega t) \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

Taking  $q_r(0) = \dot{q}_r(0) = 0$  ( $r = 1, \dots, 12$ ),  $\mu = 0.01$ , and  $\varepsilon = 0.1$ , plot  $q_1/L$  versus  $\omega t$  for  $0 \leq \omega t \leq 60$ , using 1, 3, and 12 modes. Next, using 12 modes, plot  $q_1/L$  versus  $\omega t$  for  $0 \leq \omega t \leq 60$  with  $\mu = 0.01$  and  $\varepsilon = 0.2586$ . Comment briefly on the difference between the two  $q_1/L$  versus  $\omega t$  curves obtained with 12 modes for  $\varepsilon = 0.1$  and  $\varepsilon = 0.2586$ .

**Results** Figures P14.15(a), P14.15(b)

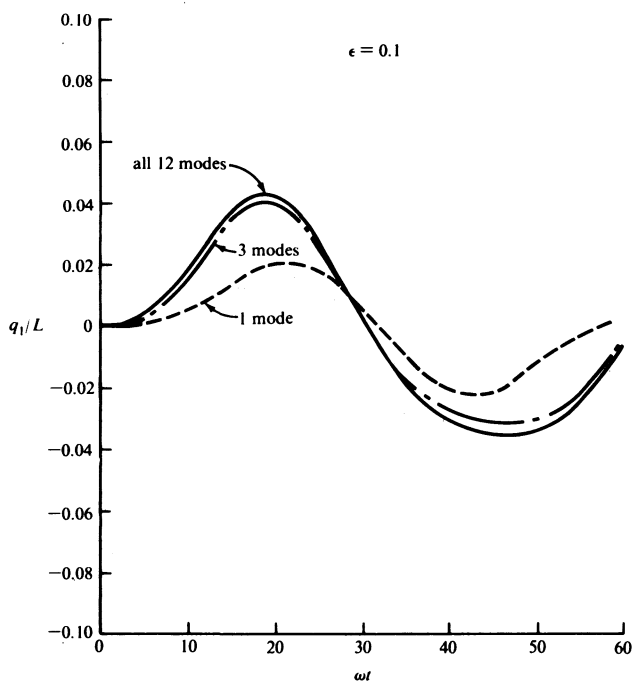


Figure P14.15(a)

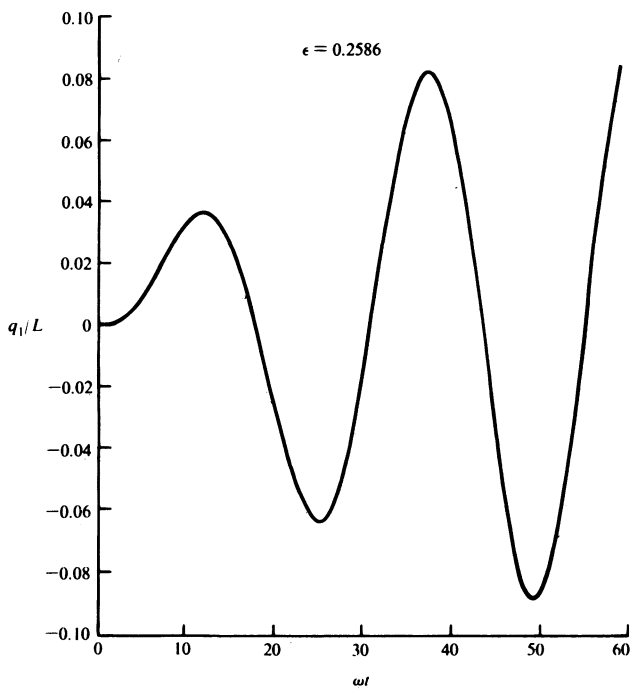


Figure P14.15(b)

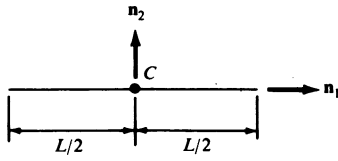


## INERTIA PROPERTIES OF UNIFORM BODIES

The information presented in Figs. A1–A28 applies variously to bodies modeled as matter distributed uniformly along a curve, over a surface, or throughout a solid. In each case, the mass center of the body under consideration is identified by the letter  $C$ , and the length of the curve (Figs. A1–A4), area of the surface (Figs. A5–A20), or volume of the solid (Figs. A21–A28) used to model the body is recorded. To distinguish figures dealing with surfaces from those involving curves or solids, shading is employed in Figs. A5–A20. For example, Figs. A18 and A24 apply to a spherical shell and a solid sphere, respectively.

The symbols  $I_1$ ,  $I_2$ , and  $I_{12}$  in Figs. A1–A28 are defined as  $I_j \triangleq \mathbf{n}_j \cdot \mathbf{I} \cdot \mathbf{n}_j$  ( $j = 1, 2$ ) and  $I_{12} \triangleq \mathbf{n}_1 \cdot \mathbf{I} \cdot \mathbf{n}_2$ , where  $\mathbf{I}$  is the central inertia dyadic (see Sec. 3.6) of the body under consideration and  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are orthogonal unit vectors shown in the figures. These unit vectors are parallel to central principal axes (see Sec. 3.8), except in the cases for which a product of inertia is reported (Figs. A5, A6, A12, A13, A17). A unit vector  $\mathbf{n}_3$ , defined as  $\mathbf{n}_3 \triangleq \mathbf{n}_1 \times \mathbf{n}_2$ , is parallel to a central principal axis in all cases; and  $I_3$ , the associated central principal moment of inertia, is given by  $I_3 = I_1 + I_2$  for the bodies in Figs. A1–A17, but *not* for those in Figs. A18–A28. In connection with Figs. A18–A25,  $I_3$  is equal to  $I_1$ ; to obtain  $I_3$  for the bodies in Figs. A26–A28, replace  $c$  with  $a$  in the expression for  $I_1$ . Finally, the symbol  $m$  denotes in every case the mass of the body under consideration.

Straight Line



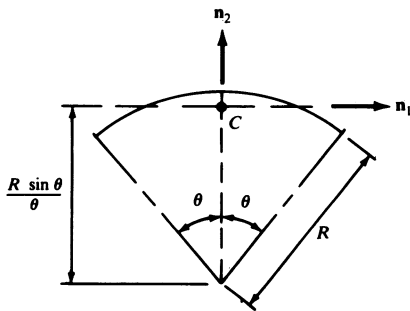
Length:  $L$

$$I_1 = 0$$

$$I_2 = \frac{mL^2}{12}$$

Figure A1

Circular Arc



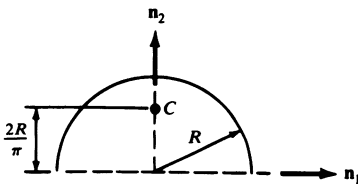
Length:  $2R\theta$

$$I_1 = \frac{mR^2}{2} \left( 1 + \frac{\sin 2\theta}{2\theta} - \frac{2 \sin^2 \theta}{\theta^2} \right)$$

$$I_2 = \frac{mR^2}{2} \left( 1 - \frac{\sin 2\theta}{2\theta} \right)$$

Figure A2

Semicircle

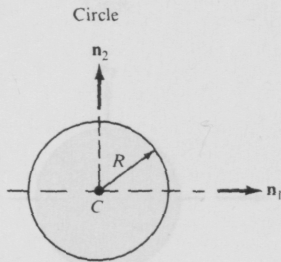


Length:  $\pi R$

$$I_1 = \frac{mR^2}{2} \left( 1 - \frac{8}{\pi^2} \right)$$

$$I_2 = \frac{mR^2}{2}$$

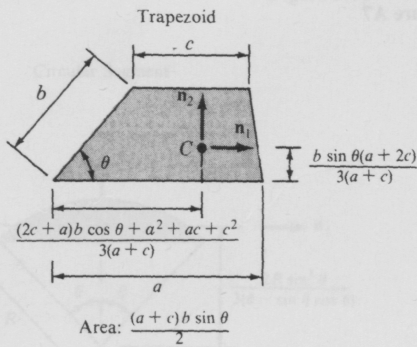
Figure A3



Length:  $2\pi R$

$$I_1 = I_2 = \frac{mR^2}{2}$$

Figure A4

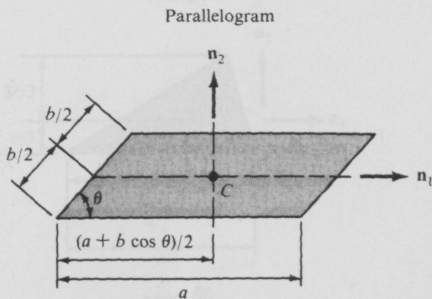


$$I_1 = \frac{m}{18} \left( \frac{b \sin \theta}{a+c} \right)^2 (a^2 + 4ac + c^2)$$

$$I_2 = \frac{m}{18(a+c)^2} [b^2(a^2 + 4ac + c^2) \cos^2 \theta - b(a^3 + 3a^2c - 3ac^2 - c^3) \cos \theta + a^4 + 2a^3c + 2ac^3 + c^4]$$

$$I_{12} = -\frac{mb \sin \theta}{36(a+c)^2} (a^2 + 4ac + c^2)(2b \cos \theta + c - a)$$

Figure A5



$$I_1 = \frac{mb^2 \sin^2 \theta}{12}$$

$$I_2 = \frac{m(a^2 + b^2 \cos^2 \theta)}{12}$$

$$I_{12} = -\frac{mb^2 \sin \theta \cos \theta}{12}$$

Figure A6

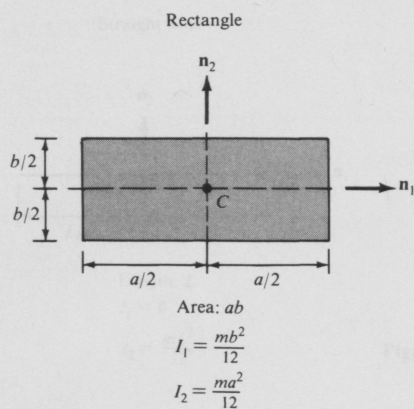


Figure A7

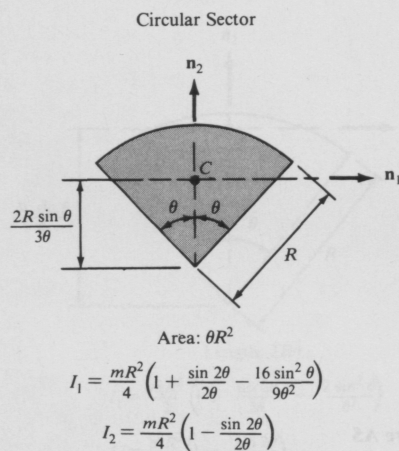


Figure A8

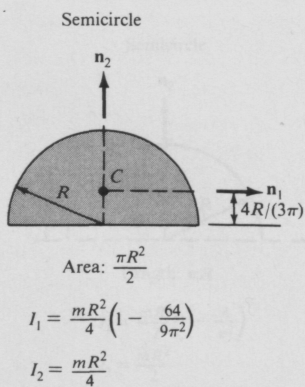
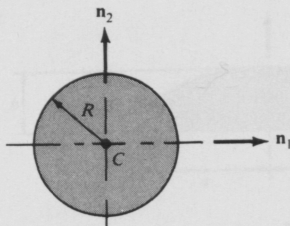


Figure A9

Circle

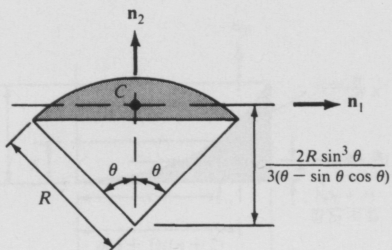


$$\text{Area: } \pi R^2$$

$$I_1 = I_2 = \frac{mR^2}{4}$$

Figure A10

Circular Segment



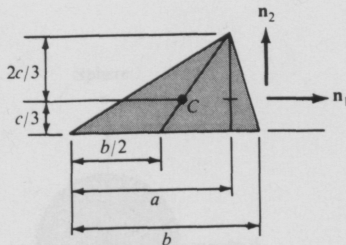
$$\text{Area: } R^2 \left( \theta - \frac{\sin 2\theta}{2} \right)$$

$$I_1 = \frac{mR^2}{4} \left[ 1 + \frac{2 \sin^3 \theta \cos \theta}{\theta - \sin \theta \cos \theta} - \frac{16 \sin^6 \theta}{9(\theta - \sin \theta \cos \theta)^2} \right]$$

$$I_2 = \frac{mR^2}{12} \left( 3 - \frac{2 \sin^3 \theta \cos \theta}{\theta - \sin \theta \cos \theta} \right)$$

Figure A11

Triangle



$$\text{Area: } \frac{bc}{2}$$

$$I_1 = \frac{mc^2}{18}$$

$$I_2 = \frac{m(a^2 - ab + b^2)}{18}$$

$$I_{12} = \frac{mc(b - 2a)}{36}$$

Figure A12

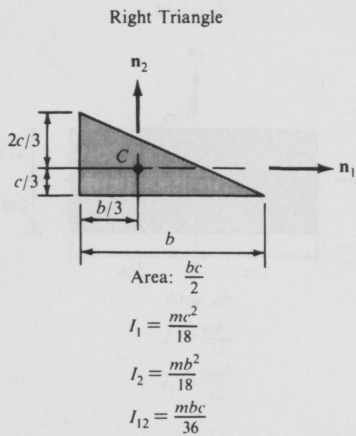


Figure A13

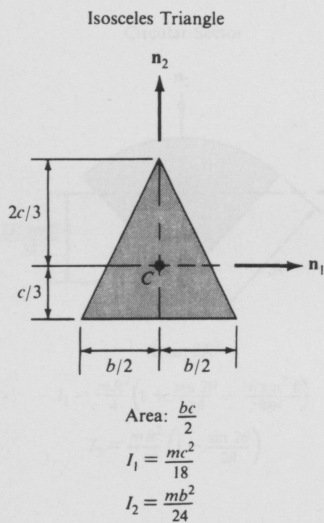


Figure A14

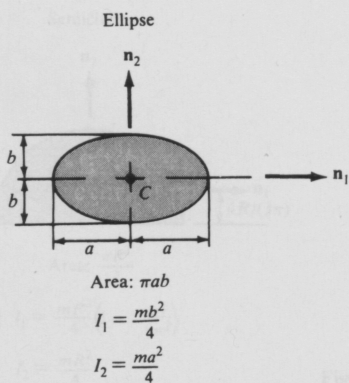
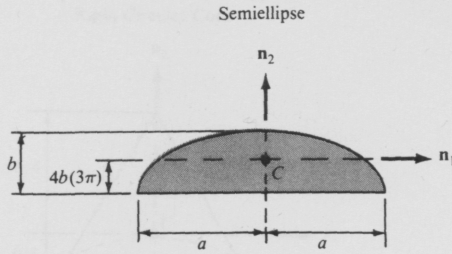


Figure A15

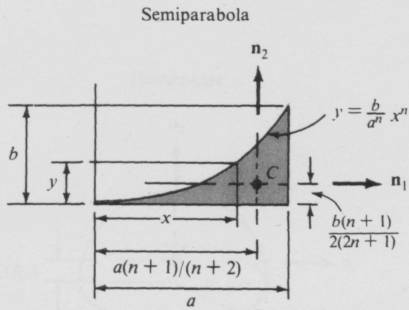


$$\text{Area: } \frac{\pi ab}{2}$$

$$I_1 = \frac{mb^2}{4} \left( 1 - \frac{64}{9\pi^2} \right)$$

$$I_2 = \frac{ma^2}{4}$$

Figure A16



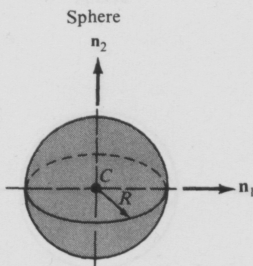
$$\text{Area: } \frac{ab}{n+1}$$

$$I_1 = \frac{mb^2(n+1)(7n^2+4n+1)}{12(3n+1)(2n+1)^2}$$

$$I_2 = \frac{ma^2(n+1)}{(n+3)(n+2)^2}$$

$$I_{12} = -\frac{mabn}{4(n+2)(2n+1)}$$

Figure A17



$$\text{Area: } 4\pi R^2$$

$$I_1 = I_2 = \frac{2mR^2}{3}$$

Figure A18



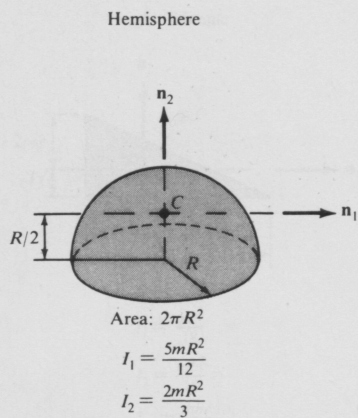


Figure A19

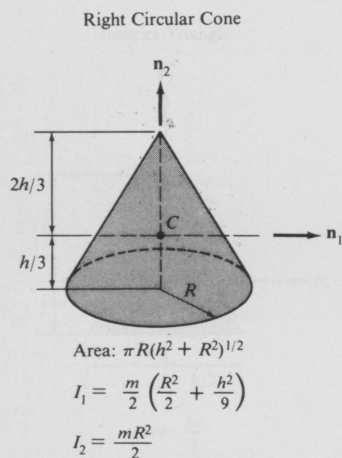


Figure A20

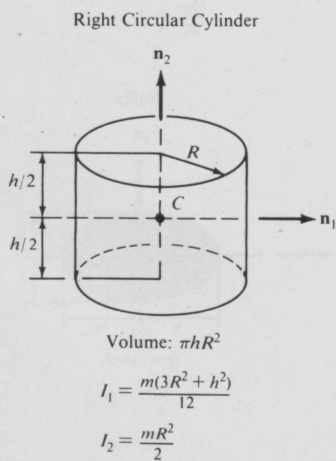
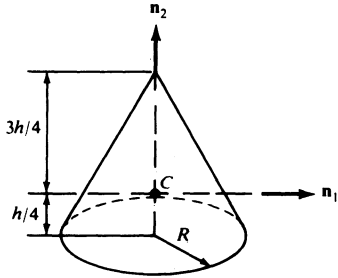


Figure A21



Right Circular Cone



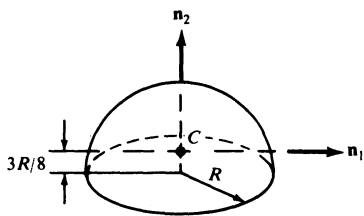
$$\text{Volume: } \frac{\pi h R^2}{3}$$

$$I_1 = \frac{3m(4R^2 + h^2)}{80}$$

$$I_2 = \frac{3mR^2}{10}$$

Figure A22

Hemisphere



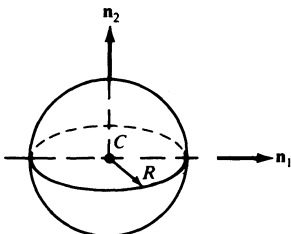
$$\text{Volume: } \frac{2\pi R^3}{3}$$

$$I_1 = \frac{83mR^2}{320}$$

$$I_2 = \frac{2mR^2}{5}$$

Figure A23

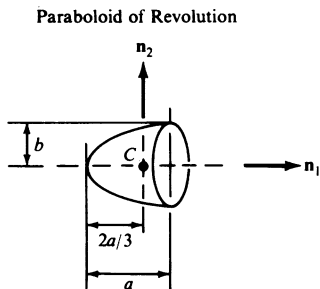
Sphere



$$\text{Volume: } \frac{4\pi R^3}{3}$$

$$I_1 = I_2 = \frac{2mR^2}{5}$$

Figure A24

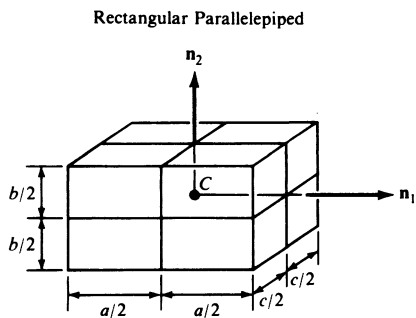


$$\text{Volume: } \frac{\pi ab^2}{2}$$

$$I_1 = \frac{mb^2}{3}$$

$$I_2 = \frac{m(a^2 + 3b^2)}{18}$$

Figure A25

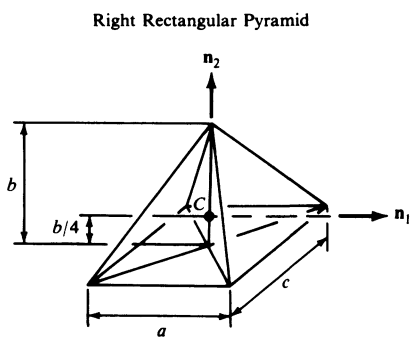


$$\text{Volume: } abc$$

$$I_1 = \frac{m(b^2 + c^2)}{12}$$

$$I_2 = \frac{m(c^2 + a^2)}{12}$$

Figure A26



$$\text{Volume: } \frac{abc}{3}$$

$$I_1 = \frac{m(3b^2 + 4c^2)}{80}$$

$$I_2 = \frac{m(c^2 + a^2)}{20}$$

Figure A27

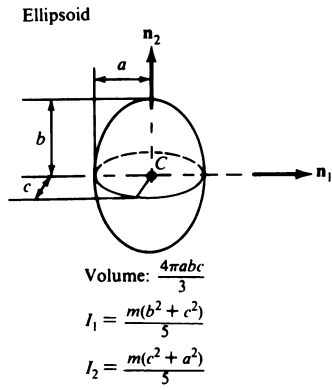


Figure A28

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the 1990s, the number of people in the UK who are employed in the public sector has increased by 1.5 million, from 2.5 million in 1980 to 4 million in 1998. The public sector has also become an important employer of women, with 5.5 million women employed in the public sector in 1998, compared with 4.5 million in 1980.

There are a number of reasons why the public sector has become an important employer of women. One reason is that the public sector has a high proportion of women in its workforce. In 1998, 88% of the public sector workforce were women, compared with 78% in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

Another reason why the public sector has become an important employer of women is that it has a high proportion of jobs that are part-time or flexible. In 1998, 28% of the public sector workforce were employed on part-time or flexible contracts, compared with 18% in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

A third reason why the public sector has become an important employer of women is that it has a high proportion of jobs that are well paid. In 1998, the average salary of a public sector employee was £18,000, compared with £15,000 in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

There are a number of other reasons why the public sector has become an important employer of women. One reason is that the public sector has a high proportion of jobs that are secure. In 1998, 88% of the public sector workforce were employed on permanent contracts, compared with 78% in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

Another reason why the public sector has become an important employer of women is that it has a high proportion of jobs that are well located. In 1998, 28% of the public sector workforce were employed in London, compared with 18% in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

A third reason why the public sector has become an important employer of women is that it has a high proportion of jobs that are well matched to women's skills. In 1998, 88% of the public sector workforce were employed in jobs that required a degree or higher qualification, compared with 78% in 1980. This is due to a number of factors, including the fact that the public sector has a high proportion of jobs that are traditionally held by women, such as teaching, nursing, and social work.

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